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Unitary spherical super-Landau models

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Abstract: A Hilbert space metric is found for the SU(2|1)-invariant 'superflag' Landau models, parametrized by integer $2N'$ and real number M, such that the Hilbert space norm is positive definite. The spectrum of these unitary super-Landau models is determined. The $M = 0$ case yields a unitary Landau model on the supersphere SU(2|1)/ U(1|1) with U(1) charge $2N = 2N' + 1$. For the generic unitary superflag model, the manifest SU(2|1) symmetry is dynamically enhanced to SU(2|2); this is the 'spherical' analog of the hidden worldline supersymmetry found previously in the planar limit.

Keywords: Superspaces, Non-Commutative Geometry, Global Symmetries, Sigma Models.

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Contents

1. Introduction

A Landau model describes the quantum mechanics of an electrically charged particle confined to a surface through which passes a constant uniform magnetic flux. In Landau's original paper the surface was planar, but this may be viewed as the $R \to \infty$ limit of a model in which the surface is a 2-sphere of radius R. In the latter case, the magnetic field can be interpreted as the field due to a magnetic monopole at the centre of a ball in $\mathbb{E}^3$ with the 2-sphere as its surface. Dirac's quantization condition then applies, so that the particle's electric charge is an integer multiple of a minimal allowed charge. We call this integer $2N$, for reasons to be explained later, and we assume it to be positive. The planar Landau model is then found by taking the limit in which $R \to \infty$ and $N \to \infty$ with $N/R^2 \equiv \kappa$ kept fixed.

In this paper we continue a study of super-Landau models [1 − 4], defined either as Landau models on homogeneous superspaces that have the 2-sphere as 'body', or as planar limits of such models. The spherical super-Landau models to be considered are those for which the superspace has a transitive action of the supergroup $\mathrm{SU}(2|1)$, which allows two possibilities. The simplest such superspace is the Riemann supersphere:

$$\mathrm{SU}(2|1)/\,\mathrm{U}(1|1) \cong \mathbb{CP}^{(1|1)} \qquad \text{[Supersphere]}\,. \tag{1.1}$$

As for the standard spherical Landau model, there is a family of superspherical Landau models indexed by a positive integer $2N$. In the planar limit one gets the superplane Landau models, indexed by the real number κ, although qualitative properties do not depend on this parameter so there is essentially only one 'superplane' model,[1] which we have investigated in detail in two previous papers [3, 4]. Excluding this case, we may set to unity the 2-sphere radius R, without loss of generality.

The generic 'spherical' super-Landau model with transitive action of SU(2|1) is a 'superflag' Landau model, for which the homogeneous superspace is

$$\mathrm{SU}(2|1)/[\mathrm{U}(1)\times \mathrm{U}(1)] \qquad \text{[Superflag]}\,. \tag{1.2}$$

Geometrically, the superflag is defined via the nested sequence of superspaces

$$\mathbb{C}^{(0|1)} \subset \mathbb{C}^{(1|1)} \subset \mathbb{C}^{(2|1)}\,. \tag{1.3}$$

Each such sequence is a point on the superflag. The supersphere is then found as the projection in which one 'forgets' the $\mathbb{C}^{(0|1)}$ superspace. If instead one 'forgets' the intermediate $\mathbb{C}^{(1|1)}$ superspace then one gets the Grassmann odd manifold $\mathrm{SU}(2|1)/\,\mathrm{U}(2)$, for which the lowest Landau level limit was considered in [9]; we shall not consider this model in detail here because it is not a 'spherical' super-Landau model. More about the geometry of flag supermanifolds may be found in [10].

[1] Here we should point out that this statement applies to the supersphere as defined above; an alternative definition yields the alternative 'supersphere' and 'superplane' Landau models studied in [5 − 7] (see [8] for a recent review).

A class of superflag models, indexed both by the positive integer $2N$ and by another continuous parameter M, was constructed in [2]. Although there is an additional complex anti-commuting coordinate ξ, as compared to the supersphere, there is also more freedom in the choice of 'kinetic' terms. In fact, there are now three separate possible SU(2|1)-invariant kinetic terms that lead to second time-derivatives in the equations of motion. One linear combination yields the Kähler sigma-model on the superflag, but the combination chosen in [2] leads to a degenerate 'metric' for which ξ has no kinetic term. The parameter M is the coefficient of a 'Wess-Zumino' term that involves the time-derivative of ξ, so the equation of motion for ξ is generally a first order differential equation, but it becomes algebraic for $M = 0$. Provided that the classical energy is non-zero, this allows ξ to be eliminated for $M = 0$, and the resulting action is precisely that of the superspherical Landau model. In the planar limit, the subtlety involving zero classical energy has no effect in the quantum theory and the $M = 0$ planar superflag model is equivalent to the superplane model [3, 4]. Here we show that a similar equivalence holds for the spherical Landau models but involving a shift of $2N$ by one unit: the $M = 0$ superflag model with charge $2N' = 2N - 1$ is equivalent to the superspherical model with charge $2N$.

Although spherical Landau models involve non-linearities absent from the planar models, they are conceptually simpler in the sense that each Landau level carries a finite-dimensional representation of the isometry group of the surface on which the particle is moving. This includes the SU(2) isometry group of the 2-sphere in all the above cases, but the SU(2) representations must combine into representations of SU(2|1) in the super-Landau models. It was shown in [1] that the lowest Landau level (LLL) of the superspherical model consists of states that span a degenerate (atypical) 'superspin' N representation of SU(2|1); this decomposes into the direct sum of a spin $(N-1/2)$ and a spin N representation of SU(2); this is one way to see why $2N$ must be an integer. A satisfactory definition of the superspherical Landau model beyond the LLL is complicated by the fact that each higher level has states of negative norm (ghosts) in the natural superspace metric [2], so the naturally defined quantum theory is not unitary. The problem is less severe for the superflag models with large positive M because the first $[2M] + 1$ levels are then ghost free [2] ($[2M]$ is the integer part of $2M$). However, there are still ghosts in the higher levels, and in all levels if $M < 0$.

In a previous paper, it was shown how this difficulty can be overcome *in the planar limit* by an alternative choice of Hilbert space norm [4]. In the planar limit, the SU(2|1) symmetry algebra gets contracted to the superalgebra ISU(1|1), and it turns out that there are two possible ISU(1|1) invariant Hilbert space norms, each associated to a choice of 'metric' operator G. The trivial choice $G = 1$ yields the indefinite Hilbert space norm but there is a second non-trivial possibility, which yields a positive definite norm for $M \leq 0$, and one can define a unitary theory for $M > 0$ by a 'dynamically chosen' mixture of the two invariant norms. The changed norm leads to a change in the operation of hermitian conjugation with the result that the new hermitian conjugates of the 'odd' charges of ISU(1|1) are shifted by odd operators that are 'new' symmetries of the model. Remarkably, these are just worldline supersymmetry charges, so the unitary 'superplane' Landau model (corresponding to the choice $M = 0$) has a hidden worldline supersymmetry (as found earlier in [7] for an alternative superplane Landau model that is apparently *quantum* equivalent to our superplane Landau model). The planar superflag model also has this worldline symmetry for $M < 0$, but it is spontaneously broken [4]. A superfield formulation of the superplane model, in which the worldline $\mathcal{N} = 2$ supersymmetry of the latter is manifest, was given in [11].

The main purpose of this paper is to present results of a similar analysis of the spherical super-Landau models. In particular we show that there are two possible SU(2|1) invariant norms on the Hilbert space of the superflag Landau model, each associated to a metric operator G. The 'trivial' choice $G = 1$ yields the indefinite Hilbert space norm but the other choice of G yields a positive Hilbert space norm provided that $-2N' - 1 < M \leq 0$, with zero norm states at $M = 0$; for other values of M one needs a 'dynamical' combination of the two norms. We solve the model in the sense that we determine the spectrum, degeneracies, and SU(2|1) representations at each level. We do the same for the supersphere; in particular, we confirm the earlier result of [1] that the LLL furnishes an irreducible superspin N representation of SU(2|1). The results agree with the $M = 0$ superflag after taking into account zero-norm states of the latter and the shift of $2N$ mentioned above, and this establishes the equivalence of these two models. For the cases in which $-2N' - 1 < M \leq 0$ we also investigate the nature of the 'hidden' symmetries that are revealed by the process described above for the planar models. For the supersphere, i.e. $M = 0$, we again find additional 'supersymmetries' but they do not form a closed algebra with the Hamiltonian, except in the planar limit; it appears likely that closure requires an infinite set of 'new' charges. Thus, the supersphere Landau model does not have a conventional worldline supersymmetry, in contrast to the superplane model. The situation for the $-2N' - 1 < M \leq 0$ superflag models is rather different, and surprising. We show that the manifest SU(2|1) symmetry of these models is enhanced to $SU(2|2)$, with the central charge being a linear function of the Landau level number.

1.1 Organization

We will start by formulating the classical superspherical Landau model. The quantum Hamiltonian is not obviously factorizable, as it is for the 'bosonic' model, but we nevertheless find an infinite set of eigenstates using covariance arguments.[2] The next step is to compute the norm of the eigenvectors. It turns out that this norm can be expressed through two analytic superfieds and that its component form is identical to a particular case of a norm considered for the superflag Landau model in [2]. In this way it is recognized that the redefinition of the norm needed for a unitary superspherical Landau model is a particular case of the redefinition needed for the superflag Landau model.

We then turn to the superflag model, reviewing results of [2]. There is an additional anti-commuting variable in comparison to the superspherical model, and this leads to 'extended' superfields upon quantization. For all Landau levels, the eigenvectors are expressed

[2] Our method, which follows the spirit of [12] and [13], can also be applied to other models and we reproduce in an appendix the results of Karabali and Nair [14] for the $\mathbb{CP}^n$ models, with the advantage that the eigenvectors are found explicitly without using knowledge of the Wigner functions for SU$(n + 1)$.

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in terms of the components of a single extended analytic superfield. We determine the action of the SU(2|1) charges in this analytic subspace, and diagonalize the natural superspace metric within it. This allows us to construct the metric operator that ensures a positive definite norm. As explained above, this can lead to the appearance of new 'hidden' symmetries, and we show that the manifest SU(2|1) symmetry is enhanced to $SU(2|2)$ when $-2N' - 1 < M < 0$.

Finally, we discuss the relationships between the supersphere and superflag models using the geometric language of non-linear realizations and covariant derivatives on the corresponding supercosets [2]. In particular, we show that the superflag Hamiltonian of ref. [2] and that of the supersphere considered here are two particular cases of a more general second-order covariant differential operator defined on the full superflag manifold $\mathrm{SU}(2|1)/[\mathrm{U}(1) \times \mathrm{U}(1)]$, and that each is recovered after imposing appropriate covariant conditions on the superfield wave functions. The covariant approach makes explicit the quantum equivalence of the $2N'$ superflag model at $M = 0$ with the $2N$ supersphere model when $2N' = 2N - 1$.

The main conclusions of this paper, taken together with our earlier work on super-Landau models, is summarized in the final section.

2. The superspherical Landau model

We begin with a presentation of some facts about SU(2|1) and the supersphere. We then construct the classical superspherical Landau model, solve the quantum model by determining the energy eigenstates and their eigenvalues. We conclude this section with a discussion of the Hilbert space norm, noting the problem of ghosts, which will be resolved following our results for the more general superflag model.

2.1 SU(2|1)

The Lie superalgebra $su(2|1)$ is spanned by even charges $(F, J_3, J_\pm)$, satisfying the commutation relations of U(2), and a U(2) doublet of odd charges (Π, Q); we write the complex conjugate charges as $(\Pi^\dagger, Q^\dagger)$ since we plan to realize this algebra in terms of operators for which $(\Pi^\dagger, Q^\dagger)$ are the hermitian conjugates of (Π, Q). The non-zero commutators of the even charges are

$$[J_+, J_-] = 2J_3\,, \qquad [J_3, J_\pm] = \pm J_\pm\,. \tag{2.1}$$

The non-zero commutators of the odd generators with the even generators are

$$\begin{aligned} [J_+, \Pi] &= iQ\,, & [J_-, Q] &= -i\Pi\,, \\ [J_3, \Pi] &= -\frac{1}{2}\Pi\,, & [J_3, Q] &= \frac{1}{2}Q\,, \\ [F, \Pi] &= -\frac{1}{2}\Pi\,, & [F, Q] &= -\frac{1}{2}Q \end{aligned} \tag{2.2}$$

and

$$\left[J_-, \Pi^\dagger\right] = iQ^\dagger\,, \qquad \left[J_+, Q^\dagger\right] = -i\Pi^\dagger\,,$$

$$
\begin{aligned}
\left[J_3, \Pi^\dagger\right] &= \frac{1}{2}\Pi^\dagger\,, & \left[J_3, Q^\dagger\right] &= -\frac{1}{2}Q^\dagger\,,\\
\left[F, \Pi^\dagger\right] &= \frac{1}{2}\Pi^\dagger\,, & \left[F, Q^\dagger\right] &= \frac{1}{2}Q^\dagger\,,
\end{aligned}
\tag{2.3}
$$

which show that (Π, Q) and $(\Pi^\dagger, Q^\dagger)$ are SU(2) doublets of charge $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. Finally, the non-zero anti-commutators of the odd charges are

$$
\begin{aligned}
\left\{\Pi, \Pi^\dagger\right\} &= -J_3 + F\,, & \left\{Q, Q^\dagger\right\} &= J_3 + F\,,\\
\left\{\Pi, Q^\dagger\right\} &= iJ_-\,, & \left\{\Pi^\dagger, Q\right\} &= -iJ_+\,.
\end{aligned}
\tag{2.4}
$$

As $su(2|1)$ is a rank two superalgebra, it has a quadratic and a cubic Casimir. The quadratic Casimir is

$$
C_2 = \frac{1}{2}\{J_+, J_-\} + J_3^2 - F^2 - \frac{1}{2}\left[\Pi, \Pi^\dagger\right] - \frac{1}{2}\left[Q, Q^\dagger\right]\,,
\tag{2.5}
$$

The cubic Casimir operator is

$$
\begin{aligned}
C_3 = {} & \frac{i}{2}J_+\left[Q^\dagger, \Pi\right] - \frac{i}{2}\left[\Pi^\dagger, Q\right]J_- + \frac{1}{2}J_3\left(\left[Q, Q^\dagger\right] - \left[\Pi, \Pi^\dagger\right]\right)\\
& -\frac{1}{2}F\left(\left[\Pi, \Pi^\dagger\right] + \left[Q, Q^\dagger\right]\right) + 2C_2F - \Pi^\dagger\Pi - QQ^\dagger.
\end{aligned}
\tag{2.6}
$$

2.2 The supersphere

The Riemann supersphere $\mathbb{CP}^{(1|1)} \cong \mathrm{SU}(2|1)/\,\mathrm{U}(1|1)$ is a complex supermanifold with complex coordinates

$$
Z^A = \left(Z^0, Z^1\right) = (z, \zeta)\,, \qquad \bar{Z}^{\bar{B}} = \left(\bar{Z}^0, \bar{Z}^1\right) = \left(\bar{z}, \bar{\zeta}\right)\,,
\tag{2.7}
$$

where z is a complex coordinate of the Riemann sphere, with complex conjugate $\bar{z}$, and ζ is its anti-commuting partner, with complex conjugate $\bar{\zeta}$. The SU(2|1) transformations of these complex coordinates are analytic and are generated by the following differential operators

$$
\begin{aligned}
F &= \frac{1}{2}\zeta\partial_\zeta\,, & J_3 &= z\partial_z + \frac{1}{2}\zeta\partial_\zeta\,,\\
J_- &= -i\partial_z\,, & J_+ &= -i\left(z^2\partial_z + z\zeta\partial_\zeta\right)\,,\\
\Pi &= \partial_\zeta\,, & \Pi^\dagger &= -\zeta z\partial_z\,,\\
Q &= z\partial_\zeta\,, & Q^\dagger &= \zeta\partial_z\,.
\end{aligned}
\tag{2.8}
$$

The notation suggests that $(\Pi^\dagger, Q^\dagger)$ may be interpreted as hermitian conjugates of (Π, Q), and this is a correct interpretation in the context of the Hilbert space norm for the superspherical Landau model that we will discuss below.

The infinitesimal SU(2|1) transformations of the coordinates are found from

$$
\delta Z^A = i\left[\lambda J_3 + \mu F + \varepsilon J_- + \bar{\varepsilon}J_+ - i\epsilon^1\Pi - i\bar{\epsilon}_1\Pi^\dagger + i\epsilon^2 Q + i\bar{\epsilon}_2 Q^\dagger\,, Z^A\right]\,,
\tag{2.9}
$$

where λ and μ are real, ε is complex with complex conjugate $\bar{\varepsilon}$, and (ϵ^1, ϵ^2) are complex anti-commuting parameters with complex conjugates $(\bar{\epsilon}_1, \bar{\epsilon}_2)$. One finds that

$$\begin{aligned} \delta z &= i\lambda z + \varepsilon + \bar{\varepsilon} z^2 - (\bar{\epsilon}_2 + z\bar{\epsilon}_1)\,\zeta\,, \\ \delta\zeta &= \frac{i}{2}\,(\lambda+\mu)\,\zeta + \epsilon^1 - \epsilon^2 z + \bar{\varepsilon} z\,\zeta\,. \end{aligned} \tag{2.10}$$

The complex conjugate expressions give the infinitesimal SU(2|1) transformations of $(\bar{z}, \bar{\zeta})$.

The Riemann supersphere is not only a complex supermanifold but also a Kähler supermanifold, with Kähler 2-form

$$\mathcal{F} = 2i\, dZ^A \wedge d\bar{Z}^{\bar{B}}\, \partial_{\bar{B}} \partial_A \mathcal{K}\,, \tag{2.11}$$

where

$$\mathcal{K} = \log\left(1 + z\bar{z} + \zeta\bar{\zeta}\right) \tag{2.12}$$

is the Kähler potential, which is real because the usual convention for complex conjugation of products of anti-commuting variables implies that $(\partial_\zeta)^* = -\partial_{\bar{\zeta}}$, and hence that

$$(\partial_{\bar{B}} \partial_A \mathcal{K})^* = (-1)^{a+b}\,(\partial_{\bar{B}} \partial_A \mathcal{K})\,. \tag{2.13}$$

Here a is the Grassmann parity associated with the A or $\bar{A}$ index; i.e. $a = 0$ for $A = 0$ and $\bar{A} = 0$, and $a = 1$ for $A = 1$ and $\bar{A} = 1$ (to avoid ambiguities with this simplified notation, one must arrange for all barred indices to have letters that differ from those of unbarred indices, but this restriction is easily accommodated).

The Kähler 2-form may be written locally as $\mathcal{F} = d\mathcal{A}$, where

$$\mathcal{A} = -i\left(dZ^A \partial_A - d\bar{Z}^{\bar{B}} \partial_{\bar{B}}\right) \mathcal{K} \equiv dZ^A \mathcal{A}_A + d\bar{Z}^{\bar{B}} \mathcal{A}_{\bar{B}} \tag{2.14}$$

is the Kähler connection. The Kähler connection transforms like a U(1) gauge potential under a Kähler gauge transformation $\mathcal{K} \to \mathcal{K} + f + \bar{f}$ for any analytic function f with complex conjugate $\bar{f}$, so $\mathcal{F}$ is Kähler gauge invariant. This implies that it is also SU(2|1) invariant because the SU(2|1) transformation of the Kähler potential is

$$\delta\mathcal{K} = \bar{\varepsilon} z + \varepsilon \bar{z} + \epsilon^1 \bar{\zeta} - \bar{\epsilon}_1 \zeta\,, \tag{2.15}$$

which is a Kähler gauge transformation.

The Kähler metric of the Riemann supersphere is

$$dZ^A d\bar{Z}^{\bar{B}}\, g_{\bar{B}A} = dZ^A d\bar{Z}^{\bar{B}}\, \partial_{\bar{B}} \partial_A \mathcal{K}\,. \tag{2.16}$$

It is manifestly Kähler gauge invariant, and hence SU(2|1) invariant. Before proceeding we record, for future use, the components of the metric and inverse metric. The metric components are

$$\begin{aligned} g_{\bar{z}z} &= \frac{1+\zeta\bar{\zeta}}{\left(1+z\bar{z}+\zeta\bar{\zeta}\right)^2}\,, & g_{\bar{z}\zeta} &= -\frac{z\bar{\zeta}}{(1+z\bar{z})^2}\,, \\ g_{\bar{\zeta}z} &= \frac{\bar{z}\zeta}{(1+z\bar{z})^2}\,, & g_{\bar{\zeta}\zeta} &= \frac{1}{1+z\bar{z}}\,. \end{aligned} \tag{2.17}$$

The inverse metric components are

$$g^{z\bar{z}} = (1 + z\bar{z})\left(1 + z\bar{z} + \zeta\bar{\zeta}\right), \qquad g^{z\bar{\zeta}} = (1 + z\bar{z})\, z\bar{\zeta}\,,$$
$$g^{\zeta\bar{z}} = -(1 + z\bar{z})\,\bar{z}\zeta\,, \qquad g^{\zeta\bar{\zeta}} = 1 + z\bar{z}\left(1 - \zeta\bar{\zeta}\right). \tag{2.18}$$

The metric $g_{\bar{B}A}$ and its inverse $g^{A\bar{B}}$ are related by the conditions

$$g^{A\bar{B}} g_{\bar{B}C} = \delta^{A}{}_{C}\,, \qquad g_{\bar{B}C}\, g^{C\bar{A}} = \delta_{\bar{B}}{}^{\bar{A}}\,. \tag{2.19}$$

2.3 The model

The classical Lagrangian of the superspherical Landau model is

$$L = \dot{Z}^A \dot{\bar{Z}}^{\bar{B}} g_{\bar{B}A} + N\left(\dot{Z}^A \mathcal{A}_A + \dot{\bar{Z}}^{\bar{B}} \mathcal{A}_{\bar{B}}\right), \tag{2.20}$$

where the overdot indicates differentiation with respect to an independent variable, which we interpret as time. Observe that L is real as a consequence of (2.13). The SU(2|1) variation of this Lagrangian is a total time derivative, for any real number N. As mentioned in the Introduction, the quantum theory requires $2N$ to be an integer, which can be interpreted as the particle's electric charge.

We will proceed directly to the Hamiltonian form of the Lagrangian,

$$L = \dot{Z}^A P_A + \dot{\bar{Z}}^{\bar{B}} P_{\bar{B}} - (P_A - N\mathcal{A}_A)\, g^{A\bar{B}} \left(P_{\bar{B}} - N\mathcal{A}_{\bar{B}}\right), \tag{2.21}$$

where the inverse metric is defined in (2.18), (2.19) and the conjugate momenta are

$$P_A = (p_z, -i\pi_\zeta)\,, \qquad P_{\bar{B}} = \left(p_{\bar{z}}, -i\pi_{\bar{\zeta}}\right). \tag{2.22}$$

Here, $p_{\bar{z}}$ is the complex conjugate of p_z and π_ζ is the complex conjugate of $\pi_{\bar{\zeta}}$; the factors of $-i$ are needed for this to be the case as a consequence of the rule for complex conjugation of products of anti-commuting variables, and this has the consequence that

$$(P_A)^* = (-1)^a\, P_{\bar{A}}\,. \tag{2.23}$$

Since the inverse metric behaves in the same way as the metric under complex conjugation, one sees that the new Lagrangian, in Hamiltonian form, is real, and one may verify that elimination of the momenta returns us to the Lagrangian (2.20). We may now read off the classical Hamiltonian, which we rewrite as

$$H_{\text{class}} = (-1)^{a(a+b)}\, g^{A\bar{B}} \left(P_A - N\mathcal{A}_A\right)\left(P_{\bar{B}} - N\mathcal{A}_{\bar{B}}\right). \tag{2.24}$$

To quantize, we make the replacements

$$p_z \to -i\partial_z\,, \qquad p_{\bar{z}} \to -i\partial_{\bar{z}}, \qquad \pi_\zeta \to \partial_\zeta\,, \qquad \pi_{\bar{\zeta}} \to \partial_{\bar{\zeta}}\,, \tag{2.25}$$

which imply

$$P_A \to -i\partial_A\,, \qquad P_{\bar{B}} \to -i\partial_{\bar{B}}\,. \tag{2.26}$$

This yields the quantum Hamiltonian

$$H = -\,(-1)^{a(a+b)}\, g^{A\bar{B}} \nabla_A^{(N)} \nabla_{\bar{B}}^{(N)}\,, \tag{2.27}$$

where

$$\nabla_A^{(N)} = \partial_A - N\,(\partial_A \mathcal{K})\,, \qquad \nabla_{\bar{B}}^{(N)} = \partial_{\bar{B}} + N\,(\partial_{\bar{B}} \mathcal{K})\,. \tag{2.28}$$

These covariant derivatives have the super-commutator

$$\nabla_{\bar{B}}^{(N)} \nabla_A^{(\tilde{N})} - (-1)^{ab}\, \nabla_A^{(\tilde{N})} \nabla_{\bar{B}}^{(N)} = -\left(N + \tilde{N}\right) g_{\bar{B}A}\,, \tag{2.29}$$

with all other super-commutators equal to zero. For further use, we present here the explicit expressions for $\nabla_A^{(N)}, \nabla_{\bar{B}}^{(N)}$:

$$\begin{aligned} \nabla_z^{(N)} &= \partial_z - N\,\frac{\bar{z}}{1+z\bar{z}+\zeta\bar{\zeta}}\,, & \nabla_{\bar{z}}^{(N)} &= \partial_{\bar{z}} + N\,\frac{z}{1+z\bar{z}+\zeta\bar{\zeta}}\,, \\ \nabla_\zeta^{(N)} &= \partial_\zeta - N\,\frac{\bar{\zeta}}{1+z\bar{z}+\zeta\bar{\zeta}}\,, & \nabla_{\bar{\zeta}}^{(N)} &= \partial_{\bar{\zeta}} - N\,\frac{\zeta}{1+z\bar{z}+\zeta\bar{\zeta}}\,. \end{aligned} \tag{2.30}$$

The SU(2|1) invariance of the model can be made manifest by writing the Hamiltonian operator in terms of the Casimir operators. One finds that

$$H = C_2\,. \tag{2.31}$$

2.4 The spectrum

The energy levels of the Landau model on the sphere may be found exactly, e.g. using a factorization method. Although it is not clear to us how to apply this method to supersphere, the 'supersymmetrization' will obviously expand the SU(2) representation content at each level to some representation of SU(2|1). Moreover, the lowest Landau level (LLL) is known from earlier work [1]; in the present context, in which we have chosen an operator ordering such that the ground state energy is zero, the LLL wave functions are components of a superfield $\Psi_0^{(N)}$, satisfying the analyticity constraint

$$\nabla_{\bar{B}}^{(N)} \Psi_0^{(N)} = 0\,, \tag{2.32}$$

and they carry an irreducible superspin N representation of SU(2|1) that decomposes into the reducible $(N-1/2)\oplus N$ representation of SU(2). More generally, the energy eigenvalues are

$$E_\ell = C_2(\ell) = \ell\,(\ell + 2N) \tag{2.33}$$

for non-negative integer ℓ, and the states in the ℓth Landau level, for $\ell > 0$, have superfield wave functions of the form

$$\Psi_\ell^{(N)} = \nabla_{A_1}^{(N+1)} \cdots \nabla_{A_\ell}^{(N+2\ell-1)} \Phi^{A_\ell \dots A_1}\,, \tag{2.34}$$

where the superfield $\Phi^{A_\ell \cdots A_1}$ is totally graded symmetric in its ℓ indices and satisfies the analyticity condition

$$\nabla_{\bar{B}}^{(N)} \Phi^{A_\ell \dots A_1} = 0\,. \tag{2.35}$$

The graded symmetry means that Φ has only two independent components, which we may take to be

$$\Phi^{z\ldots z} \equiv \Phi_{\ell}^{(+)}\,, \qquad \Phi^{z\ldots\zeta} = \Phi_{\ell}^{(-)}\,. \tag{2.36}$$

It follows that

$$\Psi_{\ell}^{(N)} = \Psi_{(+)\ell}^{(N)} + \Psi_{(-)\ell}^{(N)}\,, \tag{2.37}$$

where the two independent superfields $\Psi_{(\pm)\ell}^{(N)}$ are given by

$$\Psi_{(+)\ell}^{(N)} = \nabla_z^{(N+1)} \ldots \nabla_z^{(N+2\ell-1)}\, \Phi_{\ell}^{(+)} \tag{2.38}$$

and

$$\Psi_{(-)\ell}^{(N)} = \left[\sum_{p=1}^{\ell} \nabla_z^{(N+1)} \ldots \nabla_\zeta^{(N+2p-1)} \ldots \nabla_z^{(N+2\ell-1)}\right] \Phi_{\ell}^{(-)}\,. \tag{2.39}$$

The LLL is exceptional in that only the (+) component is defined, and this is the ground state wave function that we called $\Psi_0^{(N)}$. In general, both of the $\Psi_{(\pm)}$ components will carry an irreducible representation of SU(2|1), so only the LLL has a representation carried by a single analytic superfield. We arrived at this result using insights gained from earlier studies of the planar limit, and by analogy with the $\mathbb{CP}^2$ Landau model, which we discuss in an appendix. Here we shall verify the result for the first two levels, which is the beginning of a general inductive argument that we will not present but which should become clear.

At $\ell = 1$ we have the superfield wave function

$$\Psi_1^{(N)} = \nabla_C^{(N+1)} \Phi^C\,. \tag{2.40}$$

After acting with H on this wave function, we move the $\nabla_{\bar B}^{(N)}$ derivative to the right, where it annihilates Φ^C, but we pick up a super-commutator term, which we simplify using (2.29). The result is

$$H\Psi_1^{(N)} = (2N+1) g^{a\bar B} \nabla_A^{(N)} g_{\bar B C} \Phi^C\,. \tag{2.41}$$

Now we use the identity

$$(-1)^{a(a+b)}\, g^{A\bar B} \nabla_A^{(N)} g_{\bar B C} = \nabla_C^{(N+1)}\,, \tag{2.42}$$

which itself is a consequence of the identity

$$(-1)^{a(a+b)}\, g^{A\bar B}\, (\partial_A\, g_{\bar B C}) = -\partial_C\, \mathcal{K}\,. \tag{2.43}$$

The result is that $\Psi_1^{(N)}$ is an eigenfunction of H with energy eigenvalue $(2N+1)$.

At $\ell = 2$ we have the superfield wave function

$$\Psi_2^{(N)} = \nabla_D^{(N+1)} \nabla_C^{(N+3)} \Phi^{CD}\,. \tag{2.44}$$

After acting with H on this superfield we again move $\nabla_{\bar B}^{(N)}$ to the right, where it annihilates the chiral superfield Φ, but we now pick up two super-commutator terms. Simplifying these with (2.29), we find that

$$\begin{aligned} H\Psi_{(+)2}^{(N)} &= (-1)^{a(a+b)}\, (2N+1)\, g^{A\bar B} \nabla_A^{(N)} g_{\bar B D} \nabla_C^{(N+3)}\, \Phi^{CD} \\ &\quad + (-1)^{a(a+b)+bd}\, (2N+3)\, g^{A\bar B} \nabla_A^{(N)} \nabla_D^{(N+1)} g_{\bar B C}\, \Phi^{CD}\,. \end{aligned} \tag{2.45}$$

Now we use the identity

$$(-1)^{bc}\,\nabla^{(N+1)}_{(C}\,g_{\bar{B}D)} \equiv g_{\bar{B}(C}\nabla^{(N+3)}_{D)}\,, \tag{2.46}$$

where the brackets indicate graded symmetrization in the unbarred indices, to rewrite (2.45) as

$$H\Psi^{(N)}_{(+)2} = (-1)^{a(a+b)}\,(4N+4)\,g^{A\bar{B}}\nabla^{(N)}_{A}g_{\bar{B}D}\nabla^{(N+3)}_{C}\,\Phi^{CD}\,. \tag{2.47}$$

Then, using (2.42), we confirm that $\Psi^{(N)}_{2}$ is an eigenfunction of H with energy eigenvalue $(4N+4)$. No new identities are needed to repeat these steps at higher levels, and the result for the ℓth level may be obtained by induction. In section 5 we shall reproduce the same spectrum in an equivalent manifestly SU(2|1) covariant approach based on the standard non-linear realizations definition of covariant derivatives on (super)cosets [2].

We conclude this section with a comment. Observe that in all the above formulas the derivatives $\nabla^{(\tilde{N})}_{A}, \nabla^{(\tilde{N})}_{\bar{A}}$ $(\tilde{N} = N, N+1, \ldots)$ are defined by eqs. (2.28), (2.30): their variations under the odd part of the SU(2|1) coordinate transformations (2.10) (and the conjugate ones)[3] are

$$\begin{aligned}
\delta\nabla^{(\tilde{N})}_{z} &= (\bar{\epsilon}_1\zeta)\nabla^{(\tilde{N})}_{z} + \epsilon^2\nabla^{(\tilde{N})}_{\zeta}\,,\\
\delta\nabla^{(\tilde{N})}_{\bar{z}} &= -(\epsilon^1\bar{\zeta})\nabla^{(\tilde{N})}_{\bar{z}} + \bar{\epsilon}_2\nabla^{(\tilde{N})}_{\bar{\zeta}}\,,\\
\delta\nabla^{(\tilde{N})}_{\zeta} &= -(\bar{\epsilon}_2 + z\bar{\epsilon}_1)\nabla^{(\tilde{N})}_{z} - \tilde{N}\bar{\epsilon}_1\,,\\
\delta\nabla^{(\tilde{N})}_{\bar{\zeta}} &= (\epsilon^2 + \bar{z}\epsilon^1)\nabla^{(\tilde{N})}_{\bar{z}} - \tilde{N}\epsilon^1\,.
\end{aligned} \tag{2.48}$$

Now observe that the variations of $\nabla^{(\tilde{N})}_{\zeta}$ and $\nabla^{(\tilde{N})}_{\bar{\zeta}}$ contain pieces $\sim \tilde{N}$. For the chirality conditions (2.32), (2.35) to be covariant, we are led to ascribe similar terms to the transformations of the wave functions $\Psi^{(N)}_{0}$ and $\Phi^{(\pm)}_{\ell}$:

$$\begin{aligned}
\delta\Psi^{(N)}_{0} &= -N\,(\epsilon^1\bar{\zeta} + \bar{\epsilon}_1\zeta)\,\Psi^{(N)}_{0}\,,\\
\delta\Phi^{(+)}_{\ell} &= -N\,(\epsilon^1\bar{\zeta} + \bar{\epsilon}_1\zeta)\,\Phi^{(+)}_{\ell} - \ell(\bar{\epsilon}_1\zeta)\,\Phi^{(+)}_{\ell} + \ell(\bar{\epsilon}_2 + z\bar{\epsilon}_1)\,\Phi^{(-)}_{\ell}\,,\\
\delta\Phi^{(-)}_{\ell} &= -N\,(\epsilon^1\bar{\zeta} + \bar{\epsilon}_1\zeta)\,\Phi^{(-)}_{\ell} - (\ell-1)(\bar{\epsilon}_1\zeta)\,\Phi^{(-)}_{\ell} + \epsilon^2\,\Phi^{(+)}_{\ell}.
\end{aligned} \tag{2.49}$$

As expected, the functions $\Psi^{(N)}_{(\pm)\ell}$ defined in (2.38) and (2.39) are not separately covariant under the transformations (2.48) and (2.49), while the function $\Psi^{(N)}_{\ell}$ defined in (2.37) has a simple transformation law, the same as that of $\Psi^{(N)}_{0}$:

$$\delta\Psi^{(N)}_{\ell} = -N\,(\epsilon^1\bar{\zeta} + \bar{\epsilon}_1\zeta)\,\Psi^{(N)}_{\ell}\,. \tag{2.50}$$

The weight factor $\sim N$ in (2.49)–(2.50) is imaginary, so $|\Psi^{(N)}_{\ell}|^2 = \left(\Psi^{(N)}_{\ell}\right)^*\Psi^{(N)}_{\ell}$ is a genuine scalar.

[3]It suffices to consider only the transformations with odd parameters, as those with the even parameters are contained in the closure of those with the odd parameters.

2.5 Hilbert space norm

The Hilbert space has a natural SU(2|1)-invariant norm, defined as the superspace integral [1]

$$||\Psi||^2 = \int d\mu_0 \, e^{-\mathcal{K}} \, \Psi^* \Psi \,, \tag{2.51}$$

where

$$d\mu_0 = dz d\bar{z} \, \partial_\zeta \partial_{\bar{\zeta}} \,. \tag{2.52}$$

For the ground state this norm reproduces the results in [1]. For the first excited state we may simplify the norm by means of the integration by parts identity

$$\int d\mu_0 \, e^{-\mathcal{K}} \left(\nabla_A^{(N)} \Phi^A\right)^* \Theta \equiv -(-1)^a \int d\mu_0 \, e^{-\mathcal{K}} \left(\Phi^A\right)^* \left(\nabla_{\bar{A}}^{(N-1)} \Theta\right), \tag{2.53}$$

valid for arbitrary superfield Θ. Using also the super-commutator identity (2.29) and the chirality condition on Φ^C, we find that

$$||\Psi_1^{(N)}||^2 = (-1)^a \, (2N+1) \int d\mu_0 \, e^{-\mathcal{K}} \, \left(\Phi^B\right)^* \; g_{\bar{B}A} \Phi^A. \tag{2.54}$$

Similar steps may be used to simplify the norm of $\Psi_\ell^{(N)}$ for $\ell > 1$, but one now needs the identity, analogous to (2.46),

$$(-1)^{bc} \, \nabla_{(\bar{A}}^{(N+2)} g_{\bar{B})C} \equiv g_{(\bar{A}C} \nabla_{\bar{B})}^{(N)} \,, \tag{2.55}$$

where the brackets again indicate graded symmetrization, but now in the barred indices. The final result is

$$||\Psi_\ell^{(N)}||^2 = \sigma_\ell \frac{(2N+2\ell-1)!\ell!}{2N+\ell-1)!} \int d\mu_0 \, e^{-\mathcal{K}} \, \left(\Phi^{B_1 \ldots B_\ell}\right)^* g_{\bar{B}_1 A_1} \cdots g_{\bar{B}_\ell A_\ell} \, \Phi^{A_\ell \ldots A_1} \,, \tag{2.56}$$

where

$$\sigma_\ell = (-1)^{\sum_i^\ell b_i + \sum_i^{\ell-1} a_i b_{i+1}} \,. \tag{2.57}$$

In terms of the two independent chiral superfields $\Phi_\ell^{(\pm)}$, we have[4]

$$\begin{aligned} ||\Psi_\ell^{(N)}||^2 = \frac{(2N+2\ell-1)!\ell!}{(2N+\ell-1)!} \int d\mu_0 \, e^{-\mathcal{K}} \Big\{ & \left(\Phi_\ell^{(+)}\right)^* (g_{\bar{z}z})^\ell \, \Phi_\ell^{(+)} \\ & + \ell \left(\Phi_\ell^{(+)}\right)^* (g_{\bar{z}z})^{\ell-1} g_{\bar{z}\zeta} \, \Phi_\ell^{(-)} - \ell \left(\Phi_\ell^{(-)}\right)^* (g_{\bar{z}z})^{\ell-1} g_{\bar{\zeta}z} \, \Phi_\ell^{(+)} \\ & + \left(\Phi_\ell^{(-)}\right)^* \left[-\ell \, (g_{\bar{z}z})^{\ell-1} g_{\bar{\zeta}\zeta} + \ell \, (\ell-1) \, (g_{\bar{z}z})^{\ell-2} g_{\bar{z}\zeta} g_{\bar{\zeta}z}\right] \Phi_\ell^{(-)} \Big\}. \end{aligned} \tag{2.58}$$

To proceed, we solve the analyticity constraint (2.35) on the $\Phi_\ell^{(\pm)}$ superfields by writing

$$\Phi_\ell^{(\pm)} = e^{-N\mathcal{K}} \varphi_\ell^{(\pm)} \,, \tag{2.59}$$

[4]Although the $\ell = 0,1$ cases are special, and need to be considered separately, this result for $\ell \geq 2$ is also correct for $\ell = 0,1$. In particular, all terms involving $\Phi^{(-)}$ are absent for $\ell = 0$, as expected.

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where $\varphi_\ell^{(\pm)}$ are unconstrained analytic superfields, with holomorphic SU(2|1) transformations that follow from (2.49):

$$\begin{aligned} \delta\varphi_\ell^{(+)} &= -(2N+\ell)\,(\bar\epsilon_1\zeta)\,\varphi_\ell^{(+)} + \ell\,(\bar\epsilon_2 + z\bar\epsilon_1)\,\varphi_\ell^{(-)}\,, \\ \delta\varphi_\ell^{(-)} &= -(2N+\ell-1)\,(\bar\epsilon_1\zeta)\,\varphi_\ell^{(-)} + \epsilon_2\,\varphi_\ell^{(+)}\,. \end{aligned} \tag{2.60}$$

We may expand $\varphi_\ell^{(\pm)}$ in component fields as follows:

$$\varphi_\ell^{(-)} = A_\ell + \zeta\psi_\ell\,, \qquad \varphi_\ell^{(+)} = \chi_\ell + \zeta F_\ell\,. \tag{2.61}$$

If (as the notation suggests) the component functions (χ,ψ) are assumed to be Grassmann odd, and the component functions (A,F) are assumed to be Grassmann even, then Ψ will be Grassmann odd. With the reverse Grassmann parity assignments to the component functions, Ψ will have even Grassmann parity. In either of these two cases the 'Hilbert' space is actually a supervector space rather than a vector space. If, instead, all component functions are assumed to be Grassmann even then Ψ will not have a definite Grassmann parity but the Hilbert space will be a standard Hilbert space. There is no need here to choose between these alternatives as long as we are careful not to perform any re-ordering that would require us to specify one of them. Substituting for $\Phi_\ell^{(\pm)}$ in (2.58) and performing the Berezin integration, we arrive at the result

$$\begin{aligned} ||\Psi_\ell^{(N)}||^2 = \frac{(2N+2\ell-1)!\ell!}{(2N+\ell-1)!}\int\frac{dzd\bar z}{(1+z\bar z)^{2(N+\ell)+1}}\Bigg[&-\ell\,(2N+\ell)\,|A_\ell|^2 - \ell\bar\psi_\ell\psi_\ell \\ &-\ell\left(\bar\chi_\ell+\bar z\bar\psi_\ell\right)(\chi_\ell+z\psi_\ell)+\frac{2\,(N+\ell)+1}{1+z\bar z}\bar\chi_\ell\chi_\ell+|F_\ell|^2\Bigg]. \end{aligned} \tag{2.62}$$

If in this norm we substitute $2N = 2N'+1$, we get the norm found in [2] for the $M=0$ superflag Landau model with charge $2N'$. We shall study the general superflag Landau model in the following section, but this result already allows us to anticipate its equivalence at $M=0$ to the supersphere model, with a shift of the charge by one unit.

The above norm is SU(2|1) invariant, by construction, but not positive definite, so the associated quantum theory is not unitary. However, there could be an alternative SU(2|1) invariant norm that *is* positive-definite. Indeed there is, but we shall investigate this in the context of the more general superflag model since we may then specialize to $M=0$ to get a unitary superspherical Landau model. Quite apart from the fact that we will then have the main result in the context of a more general model, another reason for this approach to the problem is that computations are easier for the superflag model. This is because the additional anti-commuting variable of the classical theory becomes an additional superspace coordinate in the quantum theory, and expansion in this coordinate yields $(\pm)$ pairs of superfields of the type that we have been considering. This simplification also allows the superflag model to be solved exactly by a factorization trick.

3. The superflag Landau model

The superflag is the coset superspace SU(2|1)/[U(1) × U(1)]. It is a complex supermanifold and we may choose

$$Z^M = (z, \zeta, \xi)\,, \qquad \bar{Z}_M = (\bar{z}, \bar{\zeta}, \bar{\xi}) \tag{3.1}$$

as the complex coordinates, where (z, ζ) are the complex coordinates used previously for the supersphere, with SU(2|1) transformations (2.10), and ξ is a new complex anti-commuting coordinate with SU(2|1) transformation

$$\delta\xi = -\frac{i}{2}(\lambda - \mu)\,\xi + \epsilon^2 - \bar{\varepsilon}\zeta + (\bar{\epsilon}_1\zeta - \bar{\varepsilon}z)\,\xi\,. \tag{3.2}$$

The superflag is also a Kähler supermanifold, but the Kähler metric is *not* used in the superflag Landau model, as constructed in [2]. Instead one uses another SU(2|1)-invariant second-rank tensor field, a *degenerate* one such that there is no 'kinetic' term for the new variable ξ. Specifically, the 'kinetic' part of the Lagrangian is constructed from a complex SU(2|1)-invariant super one-form that induces a worldline one-form with the coefficient[5]

$$\omega^+ = K_2^{-1}K_1^{-\frac{1}{2}}\left\{\dot{z}\left[1 - z\xi\bar{\zeta} - K_2\xi\bar{\xi}\right] - \dot{\zeta}\left[z\bar{\zeta} + K_2\bar{\xi}\right]\right\}, \tag{3.3}$$

where

$$K_1 = 1 + \left(\bar{\zeta} + \bar{z}\bar{\xi}\right)(\zeta + z\xi) + \bar{\xi}\xi\,, \qquad K_2 = 1 + \bar{z}z + \zeta\bar{\zeta}\,. \tag{3.4}$$

In addition, the model uses the two real SU(2|1)-invariant super 2-forms

$$\begin{aligned} F_1 &= 2i dZ^M \wedge d\bar{Z}^{\bar{N}}\partial_{\bar{N}}\partial_M \log K_1 = d\mathcal{B} \\ F_2 &= -2i dZ^M \wedge d\bar{Z}^{\bar{N}}\partial_{\bar{N}}\partial_M \log K_2 = d\mathcal{A}\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \mathcal{B} &= i\left(dZ^M\partial_M - d\bar{Z}^{\bar{M}}\partial_{\bar{M}}\right)\log K_1 = dZ^M\mathcal{B}_M + d\bar{Z}^{\bar{M}}\mathcal{B}_{\bar{M}}\,, \\ \mathcal{A} &= -i\left(dZ^M\partial_M - d\bar{Z}^{\bar{M}}\partial_{\bar{M}}\right)\log K_2 = dZ^M\mathcal{A}_M + d\bar{Z}^{\bar{M}}\mathcal{A}_{\bar{M}}\,. \end{aligned} \tag{3.6}$$

The SU(2|1)-invariance of F_1 follows directly from the transformation law

$$\delta(\log K_1) = \left(\bar{\epsilon}_1\zeta - \epsilon^1\bar{\zeta}\right) + (\bar{\epsilon}_2 + z\bar{\epsilon}_1)\,\xi - \left(\epsilon^2 + \bar{z}\epsilon^1\right)\bar{\xi}\,. \tag{3.7}$$

The super 2-form F_2 is the Kähler 2-form of the supersphere model since $\log K_2$ is the Kähler potential $\mathcal{K}$ of the supersphere. Consequently, $\mathcal{A}$ is the U(1) connection used in the construction of the supersphere model; in particular, its ξ component is zero, and the non-zero components are ξ-independent.

We now have all the ingredients needed for the generalization from the supersphere Landau model to the superflag Landau model. The superflag Lagrangian is

$$L = \left|\omega^+\right|^2 + \left[\dot{Z}^M\left(N'\mathcal{A}_M + M\mathcal{B}_M\right) + c.c.\right], \tag{3.8}$$

[5]This is equivalent to the expression of [2], which is given there in different coordinates.

where N' and M are two real numbers. In the quantum theory, M remains arbitrary but $2N'$ must be an integer; we will later see that the $M = 0$ superflag model is *quantum* equivalent to the supersphere model when $2N = 2N' + 1$, but let us first consider the relation between the classical Lagrangians of these models. When $M = 0$ there are no terms involving time derivatives of ξ in (3.8), so the equation of motion of this variable is algebraic. By making explicit the ξ dependence in the Lagrangian, one finds that the ξ equation of motion, for $M = 0$, is

$$\left[\left(1-\bar\zeta\zeta\right)|\dot z|^2 + \dot z\bar z\dot{\bar\zeta}\zeta - \dot{\bar z}z\dot\zeta\bar\zeta + \dot\zeta\dot{\bar\zeta}\left(K_2-\bar\zeta\zeta\right)\right]\xi = -\dot\zeta\left[\dot{\bar z}\left(1-\bar\zeta\zeta\right)+\bar z\dot{\bar\zeta}\zeta\right]. \tag{3.9}$$

As long as $\dot z \neq 0$, one may use this equation to eliminate ξ, in which case the resulting Lagrangian is equivalent to the Lagrangian for the superspherical Landau model, with $N' = N$. However, when $\dot z = 0$, (3.9) is equivalent to

$$\dot\zeta\dot{\bar\zeta}\left[\left(K_2-\bar\zeta\zeta\right)\xi + \bar z\zeta\right] = 0\,, \tag{3.10}$$

so the solution for ξ is no longer unique but involves terms proportional to $\dot\zeta$ and $\dot{\bar\zeta}$ with *arbitrary* functions as coefficients. As we will see shortly, this feature is associated to a fermionic gauge invariance of the $M = 0$ superflag model when restricted to configurations with zero energy.

As the last topic of this subsection we note that the holomorphic superspace (z, ζ, ξ) can be extended to the following complex supermanifold

$$(z, \zeta, \xi, \widehat{\bar\xi}), \qquad \widehat{\bar\xi} = \bar\xi\, K_2 + \bar\zeta\, z\,, \tag{3.11}$$

which is still closed under the action of SU(2|1):

$$\delta\widehat{\bar\xi} = \frac{i}{2}\left(\lambda-\mu\right)\widehat{\bar\xi} + \bar\varepsilon z\,\widehat{\bar\xi} + \bar\epsilon_2 + \bar\epsilon_1\left(z + \widehat{\bar\xi}\,\zeta\right). \tag{3.12}$$

This extension of the holomorphic supersphere (z, ζ) will be exploited in section 5 where we revisit the relationships between the supersphere and superflag Landau models.

3.1 Hamiltonian

We now turn to a Hamiltonian analysis of the general superflag model. The model has four primary constraints, which occur in two complex conjugate pairs. One pair is

$$\varphi_\zeta = \mathcal{P}_\zeta + i(\bar\xi K_2 + \bar\zeta z)\mathcal{P}_z, \qquad \varphi_{\bar\zeta} = \mathcal{P}_{\bar\zeta} - i(\xi K_2 + \zeta\bar z)\mathcal{P}_{\bar z}\,, \tag{3.13}$$

where

$$\begin{aligned} \mathcal{P}_\zeta &= \pi_\zeta - iN'\mathcal{A}_\zeta - iM\mathcal{B}_\zeta, \qquad & \mathcal{P}_z &= (p_z - N'\mathcal{A}_z - M\mathcal{B}_z),\\ \mathcal{P}_{\bar\zeta} &= \pi_{\bar\zeta} - iN'\mathcal{A}_{\bar\zeta} - iM\mathcal{B}_{\bar\zeta}, \qquad & \mathcal{P}_{\bar z} &= (p_{\bar z} - N'\mathcal{A}_{\bar z} - M\mathcal{B}_{\bar z})\,. \end{aligned} \tag{3.14}$$

The other pair is

$$\varphi_\xi = \pi_\xi - iM\mathcal{B}_\xi, \qquad \varphi_{\bar\xi} = \pi_{\bar\xi} - iM\mathcal{B}_{\bar\xi}\,. \tag{3.15}$$

The Hamiltonian is

$$H_0 = K_2^2 K_1^{-1} \left[1 + \left(\bar{\zeta} + \bar{z}\bar{\xi}\right)\zeta\right]\left[1 + \bar{\zeta}\left(\zeta + z\xi\right)\right]\mathcal{P}_z\mathcal{P}_{\bar{z}}\,, \tag{3.16}$$

where the subscript is a reminder that we may add any function on phase space that vanishes on the subspace specified by the primary constraints. A remarkable feature of this Hamiltonian is that it is independent of M. When we pass to the quantum theory, this means that the energy levels are independent of M but this does not mean that the parameter M is irrelevant because it can affect the norms of the quantum states. This effect has a classical counterpart that we now explain.

A computation shows that the analytic constraint functions $(\varphi_\zeta, \varphi_\xi)$ have vanishing Poisson brackets among themselves, but that the matrix of Poisson brackets of these functions with their complex conjugates is non-zero. In fact,

$$\det\begin{pmatrix}\{\varphi_\zeta,\varphi_{\bar{\zeta}}\}_{PB} & \{\varphi_\zeta,\varphi_{\bar{\xi}}\}_{PB}\\ \{\varphi_\xi,\varphi_{\bar{\zeta}}\}_{PB} & \{\varphi_\xi,\varphi_{\bar{\xi}}\}_{PB}\end{pmatrix} = -\left(1+\zeta\bar{\zeta}\right)\left(1+K_2\,\bar{\xi}\xi\right)\left[H_0 - 4M\left(N'+M\right)\right]. \tag{3.17}$$

It follows that there is a gauge invariance on the surface in phase space with energy

$$H_0 = 4M\left(N'+M\right). \tag{3.18}$$

Indeed, if the determinant of the constraints is weakly zero then the matrix of Dirac brackets of the constraint functions is degenerate and some constraints must be 'first class', in Dirac's terminology, and according to Dirac's formalism there is a gauge invariance for each first class constraint. As the constraints are Grassmann odd in our case, the gauge invariances have Grassmann odd parameters. This generalizes the analogous result of [3] for the planar superflag model. From the analysis of the planar limit, we expect that this classical gauge invariance leads to zero-norm states in the quantum theory whenever there is an energy level with energy $4M(N'+M)$, and we confirm this below. Note, in particular, that this implies that there are zero norm ground states when $M=0$.

Before proceeding to the quantum theory we have to address a minor difficulty. The Hamiltonian H_0 does not commute, even 'weakly', with the constraints. This difficulty can be circumvented by introducing the new variables

$$\xi^1 = \zeta + z\xi\,, \qquad \xi^2 = \xi\,. \tag{3.19}$$

These were the variables used in [2], and the analog of H_0 found by using these variables commutes with the constraints. Alternatively, one can modify the Hamiltonian by adding terms proportional to the constraint functions such that the new Hamiltonian commutes, at least weakly, with the constraints. This second approach was the one adopted in [4] for the planar superflag, and we will do the same here. Specifically, we take the new Hamiltonian to be

$$H = K_2^2 K_1\left(\mathcal{P}_z + i\xi\mathcal{P}_\zeta\right)\left(\mathcal{P}_{\bar{z}} + i\xi\mathcal{P}_{\bar{\zeta}}\right). \tag{3.20}$$

It may be verified that H is weakly equivalent to H_0 but commutes (strongly) with the constraints.

3.2 Quantum theory

To pass to the quantum theory we make the replacement $P_A \to -i\partial_A$, as in (2.26), where $A = (z, \zeta)$, and we also make the replacement

$$\pi_\xi \to \partial_\xi\,, \qquad \pi_{\bar\xi} \to \partial_{\bar\xi}\,, \tag{3.21}$$

which is needed only for the second pair of constraints (3.15). The resulting Hamiltonian operator[6] is

$$H_{N'} = -K_2^2 K_1 \left(\nabla_z^{(N')} - \xi\nabla_\zeta^{(N')}\right)\left(\nabla_{\bar z}^{(N')} - \bar\xi\nabla_{\bar\zeta}^{(N')}\right), \tag{3.22}$$

where

$$\nabla_A^{(N')} = \partial_A - iN'\mathcal{A}_A\,, \qquad \nabla_{\bar A}^{(N')} = \partial_{\bar A} - iN'\mathcal{A}_{\bar A}\,. \tag{3.23}$$

Because the analytic constraint operators commute, we may quantize *à la* Gupta-Bleuler by requiring physical states to be annihilated by these operators. The result is that 'physical' wave functions must take the form

$$\Psi = K_1^M K_2^{-N'}\,\Phi\left(z, \bar z_{sh}, \zeta, \xi\right), \tag{3.24}$$

where Φ is a 'reduced' wave function that depends on $\bar z$ only through the 'shifted' coordinate

$$\bar z_{sh} = \bar z - \xi\bar\zeta - \bar z\left(\zeta + z\xi\right)\bar\zeta\,. \tag{3.25}$$

For $2N'$ an integer, which we may assume to be positive, the Hamiltonian may be diagonalized in the physical subspace, with energy eigenvalues [2]

$$E_{N'} = \ell(2N' + \ell + 1)\,, \qquad \ell = 0, 1, 2, \ldots\,. \tag{3.26}$$

The wave functions for the LLL ($\ell = 0$) is

$$\Psi^{(0)} = K_1^M K_2^{-N'}\Phi_{an}^{(0)}\left(z, \zeta, \xi\right). \tag{3.27}$$

That is, the reduced LLL wave function is an *analytic* function. The reduced wave function at all higher levels may be expressed in terms of a level ℓ *analytic* function $\Phi_{an}^{(\ell)}$ according to the formula

$$\Phi^{(\ell)} = \mathcal{D}^{2(N'+1)}\cdots\mathcal{D}^{2(N'+\ell)}\,\Phi_{an}^{(\ell)}\left(z, \zeta, \xi\right) \qquad (\ell > 0)\,, \tag{3.28}$$

where

$$\mathcal{D}^{2N'} \equiv \nabla_z^{2N'} - \xi\nabla_\zeta^{2N'} = \partial_z - \xi\partial_\zeta - \frac{2N'\,\bar z_{sh}}{1 + z\bar z_{sh}}\,. \tag{3.29}$$

As in the case of the superspherical Landau model, there is a natural SU(2|1) invariant inner product on Hilbert space defined by a superspace integral, although the superspace now has an additional complex anti-commuting coordinate. As shown in [2], this inner product is

$$\langle\Upsilon|\Psi\rangle = \int dz d\bar z\,\partial_\zeta\partial_{\bar\zeta}\partial_\xi\partial_{\bar\xi}\,K_2^{-2}\,\Upsilon^*\Psi\,. \tag{3.30}$$

[6]Operator ordering ambiguities allow the addition of a constant, which we have set to zero.

Performing the Berezin integration over *all* anti-commuting coordinates, we get an ordinary integral over the sphere with an integrand determined by the four analytic functions $(A^{(\ell)}, \psi^{(\ell)}, \chi^{(\ell)}, F^{(\ell)})$ appearing in the (ζ, ξ)-expansion of $\Phi^{(\ell)}_{an}$:

$$\Phi^{(\ell)}_{an} = A^{(\ell)} + \zeta \left[\psi^{(\ell)} + \frac{\partial_z \chi^{(\ell)}}{(2N' + 2\ell + 1)} \right] + \xi \chi^{(\ell)} + \zeta\xi\, F^{(\ell)} . \tag{3.31}$$

The net result, after integrating by parts to remove all derivatives, is that wave functions at different levels are orthogonal, while

$$\begin{aligned} ||\Psi^{(\ell)}_{N'}||^2 \equiv \langle \Psi | \Psi \rangle = \ell! \frac{(2N' + \ell + 1)!}{(2N' + 1)!} \int \frac{dz d\bar{z}}{(1 + z\bar{z})^{2(N'+\ell+1)}} \\ \times \Bigg\{ (2M - \ell) \big(2M + 2N' + \ell + 1\big)\, \bar{A}^{(\ell)} A^{(\ell)} + \bar{F}^{(\ell)} F^{(\ell)} \\ + \frac{(N' + \ell + 1)\,(2N' + 2M + \ell + 1)}{(2N' + 2\ell + 1)\,(1 + z\bar{z})} \bar{\chi}^{(\ell)} \chi^{(\ell)} \\ + (2M - \ell)\,(1 + z\bar{z})\, \bar{\psi}^{(\ell)} \psi^{(\ell)} \Bigg\} . \end{aligned} \tag{3.32}$$

This is a simplified form of the result given in [2]; the unusual expansion of (3.31) has led to a norm that is diagonal in the component functions. The finiteness of the norm (the S^2 square-integrability requirement) requires, as usual, that fields of SU(2) spin s are degree $2s$ holomorphic polynomials in z. The SU(2) spin content will be computed explicitly in the following subsection, but it is not difficult to see what the result will be. The fields $A^{(\ell)}(z)$ and $F^{(\ell)}(z)$ each have spin $s = N' + \ell$, while the fields $\chi^{(\ell)}(z)$ and $\psi^{(\ell)}(z)$ have, respectively, spins $s = N' + \ell + \frac{1}{2}$ and $s = N' + \ell - \frac{1}{2}$. This demonstrates, in particular, the equality of the numbers of fermionic and bosonic degrees of freedom at any Landau level without zero-norm states.

With the above norm, the model has ghosts. For positive M (which was the only case considered in [2]) there are ghosts whenever $\ell > 2M$ and if $2M$ is a non-negative integer then there are zero-norm states for $\ell = 2M$. This means, in particular, that the model has ghosts in this 'naive' norm for any positive M . The same is true for negative M , and in this case there are zero norm states even for $\ell = 0$.

Of course, the sign of the norm has physical relevance only for Grassmann-even component functions, and either $A^{(\ell)}$ or $\psi^{(\ell)}$ would be Grassmann-odd if we were to assume (as in [2]) that wave functions are superfields (i.e. have definite Grassmann parity). However, even in this case the above statements concerning ghosts still apply. We have been careful to allow for (i) wave functions that are superfields, in which case the 'Hilbert' space is actually a vector superspace, and (ii) wave functions for which all component fields are ordinary functions (or bundle sections), in which case the Hilbert space is a vector space. The ghost problem can be circumvented by another choice of SU(2|1)-invariant norm, but we postpone the construction of this alternative norm until we have achieved a better understanding of the action of SU(2|1) on the superflag Hilbert space.

3.3 Unitary norm

The SU(2|1) symmetry of the superflag model implies the existence of Noether charges, which become differential operators in the quantum theory, satisfying the (anti)commutation relations of SU(2|1) given in section 2.2. These differential operators acting on the whole superflag wave functions, determine a simpler set of differential operators that act on the analytic wave functions, and vice-versa since the full Noether charge operators can be recovered from the simpler 'analytic' operators that we now present. The even generators are

$$\begin{aligned} J_- &= -i\partial_z\,, \\ J_+ &= -i\left[-2\left(N'+\ell\right)z + z^2\partial_z + z\zeta\partial_\zeta - \left(\zeta + z\xi\right)\partial_\xi\right], \\ J_3 &= -\left(N'+\ell\right) + z\partial_z + \frac{1}{2}\left(\zeta\partial_\zeta - \xi\partial_\xi\right), \\ F &= 2M + N' + \frac{1}{2}\left(\zeta\partial_\zeta + \xi\partial_\xi\right). \end{aligned} \tag{3.33}$$

Note the ℓ-independence of B; for the other generators one should view ℓ as an operator (later to be called L) that takes the value ℓ in the ℓth level. The odd generators are

$$\Pi = \partial_\zeta\,, \qquad Q = z\partial_\zeta - \partial_\xi \tag{3.34}$$

and

$$\begin{aligned} \Pi^\dagger &= \left(2M + 2N' + \ell\right)\zeta - \zeta z\partial_z + \xi\left[\left(2M-\ell\right)z - \zeta\partial_\xi\right], \\ Q^\dagger &= \zeta\partial_z - \left(2M-\ell\right)\xi\,. \end{aligned} \tag{3.35}$$

These results may be compared to the expressions (2.8). In the present case, the full differential operators representing the generators $(J_+, \Pi^\dagger, Q^\dagger)$, which are determined by the simpler 'analytic' forms given above, are the Hermitian conjugates of the generators (J_-, Π, Q) in the 'naive' norm.

We are now in a position to work out the SU(2|1) representation content at each Landau level. Let us first consider the SU(2) content. We have

$$\begin{aligned} J^2 &= J_-J_+ + J_3^2 + J_3 \\ &= \left(N'+\ell+1\right)\left(N'+\ell\right) - \left(N'+\ell+\frac{1}{4}\right)\zeta\partial_\zeta \\ &\quad + \left[\zeta\partial_z + \left(N'+\ell+\frac{3}{4} - \frac{1}{2}\zeta\partial_\zeta\right)\xi\right]\partial_\xi\,. \end{aligned} \tag{3.36}$$

Now we act with this operator on the analytic wave functions of (3.31), which we may rewrite as

$$\Phi_{an}^{(\ell)} = A^{(\ell)} + \zeta\psi^{(\ell)} + \left[\xi + \frac{\zeta\partial_z}{2N'+2\ell+1}\right]\chi^{(\ell)} + \zeta\xi\,F^{(\ell)}\,. \tag{3.37}$$

We find that

$$J^2\Phi_{an}^{(\ell)} = \left(N'+\ell\right)\left(N'+\ell+1\right)A^{(\ell)} + \left(N'+\ell-\frac{1}{2}\right)\left(N'+\ell+\frac{1}{2}\right)\zeta\psi^{(\ell)}$$

$$+\left(N'+\ell+\frac{1}{2}\right)\left(N'+\ell+\frac{3}{2}\right)\left[\xi+\frac{\zeta\partial_z}{2N'+2\ell+1}\right]\chi^{(\ell)}$$
$$+\left(N'+\ell\right)\left(N'+\ell+1\right)\zeta\xi\,F^{(\ell)}\,. \tag{3.38}$$

One reads off from this result the eigenfunctions of J^2 and their eigenvalues. Acting with J_3 on the J^2 eigenfunctions we get

$$\begin{aligned}
J_3\left[A^{(\ell)}\right] &= \left(z\partial_z - N' - \ell\right)A^{(\ell)}\,,\\
J_3\left[\zeta\psi^{(\ell)}\right] &= \zeta\left(z\partial_z - N' - \ell + \frac{1}{2}\right)\psi^{(\ell)}\,,\\
J_3\left[\left(\xi+\frac{\zeta\partial_z}{2N'+2\ell+1}\right)\chi^{(\ell)}\right] &= \left(\xi+\frac{\zeta\partial_z}{2N'+2\ell+1}\right)\left(z\partial_z - N' - \ell - \frac{1}{2}\right)\chi^{(\ell)}\\
J_3\left[\zeta\xi\,F^{(\ell)}\right] &= \zeta\xi\left(z\partial_z - N' - \ell\right)F^{(\ell)}\,.
\end{aligned} \tag{3.39}$$

Putting this all together we find the following sets of $(2s+1)$ spin-s joint eigenfunctions of J^2 and J_3:

$$\begin{aligned}
&s=\left(N'+\ell\right)\;:\;z^n a_n\,, && n=0,\ldots,2N'+2\ell\,,\\
&s=\left(N'+\ell-\frac{1}{2}\right)\;:\;\zeta z^p\psi_n\,, && p=0,\ldots,2N'+2\ell-1\,,\\
&s=\left(N'+\ell+\frac{1}{2}\right)\;:\left(\xi+\frac{(q+1)\,\zeta}{2N'+2\ell+1}\right)z^q\chi_q\,, && q=0,\ldots,2N'+2\ell+1\,,\\
&s=\left(N'+\ell\right)\;:\;\zeta\xi\,z^m f_m\,, && m=0,\ldots,2N'+2\ell
\end{aligned} \tag{3.40}$$

for constants (a_m,ψ_p,χ_q,f_m).

As mentioned already, there are two separate cases in which the 'naive' norm considered so far has ghosts when $M<0$. These are (i) $2M<-2N'-1$, and (ii) $-2N'-1<2M<0$. Consider the operator

$$G_{an} = -1 + 2\xi\partial_\xi + \frac{2}{2N+2\ell+1}\,\zeta\partial_z\partial_\xi\,. \tag{3.41}$$

This commutes with J^2 and J_3, and hence with the Hamiltonian, as is clear from the alternative expression

$$G_{an} = \frac{1}{2N'+2\ell+1}\left[2J^2 + 2\left(F-2M+\ell\right)^2 - \left(2N'+2\ell+1\right)^2\right]. \tag{3.42}$$

It also has the property that

$$G_{an}^2 \equiv 1\,. \tag{3.43}$$

As explained in [4], the same properties hold for the corresponding 'full' operator G, so each of the eigenstates listed above has a definite 'G-parity'. By inspection, one sees that for

$$-2N'-1<2M<0\,, \tag{3.44}$$

the positive (negative) norm eigenstates have positive (negative) G-parity, and therefore that the G is the 'metric operator' for M in the above range, in the sense that the new norm

$$|||\Psi|||^2 \equiv \langle\Psi|G\Psi\rangle \tag{3.45}$$

is positive definite; we refer to [4] for details of the formalism. In the planar limit, this range extends to all negative M, so we should expect the planar limit of G_{an} to be the $M < 0$ metric operator of the planar superflag found in [4], and this is indeed the case. For $M = 0$ there are zero-norm states, as in the planar limit, but still no negative-norm states. This allows us to redefine the states in a 'physical' Hilbert space to be equivalence classes of states in the original Hilbert space in which two states that differ by a zero-norm state are considered equivalent.

Now consider the operator

$$\tilde{G}_{an} = 1 - 8\left(F - 2M - N'\right) + 8\left(F - 2M - N'\right)^2 . \tag{3.46}$$

It is manifest that $\tilde{G}_{an}$ commutes with the Hamiltonian, and hence the same is true of $\tilde{G}$. One may verify that $\tilde{G}_{an}^2 \equiv 1$, so that the eigenstates listed above also have a definite $\tilde{G}$-parity. Inspection shows that when $2M < -2N' - 1$ the states with positive (negative) norm have (positive) negative $\tilde{G}$-parity. The operator $\tilde{G}$ is therefore a 'metric' operator for $2M < -2N' - 1$, which is a range that has no counterpart in the planar limit. As in the planar limit, the metric operator for $M > 0$ is a more-complicated 'dynamical' one, depending on the level. We skip the details of this case.

4. Hidden symmetries

We know that there is hidden worldline supersymmetry of the *planar* super-Landau models, for $M \leq 0$. This implies the existence of some enlarged supersymmetry algebra for the spherical super-Landau models, and we now aim to investigate this. For simplicity, we now place M in the range for which the metric operator defining the unitary models is the operator G defined by (3.41). As we have seen, this means that M should satisfy (3.44) but, as we have also seen, we may allow $M = 0$ too. In other words, we now restrict M such that

$$-2N' - 1 < 2M \leq 0 . \tag{4.1}$$

Now, let $\mathcal{O}$ be some operator that commutes with the Hamiltonian, and hence generates some symmetry of the model under investigation, and let $\mathcal{O}^\dagger$ be its hermitian conjugate with respect to the 'naive', and non-positive, Hilbert space norm. Then its hermitian conjugate with respect to the positive Hilbert space norm is (recall that $G^2 \equiv 1$)

$$\mathcal{O}^\ddagger \equiv G\mathcal{O}^\dagger G = \mathcal{O}^\dagger + G\mathcal{O}_G^\dagger , \tag{4.2}$$

where

$$\mathcal{O}_G \equiv [G, \mathcal{O}] \tag{4.3}$$

is another operator that commutes with the Hamiltonian. Note that

$$(\mathcal{O}_G)^\ddagger = \left[G, \mathcal{O}^\dagger\right] = -\left[G, \mathcal{O}\right]^\dagger = -\left(\mathcal{O}_G\right)^\dagger \equiv -\mathcal{O}_G^\dagger\,. \tag{4.4}$$

Symmetry generators that do not commute with G thus generate, in general, additional symmetries that are 'hidden' in the sense that their existence was not built into the construction of the model. For the superflag model, it is the odd generators that fail to commute with G, and this leads to the following new symmetry generators

$$\begin{aligned}
\Pi_G &= -\frac{2}{2N'+2\ell+1}\,\partial_\xi\partial_z\,,\\
\Pi_G^\ddagger &= \frac{4M-2\ell}{2N'+2\ell+1}\left[\zeta\left(1+z\partial_z\right)+\left(2N'+2\ell+1\right)z\xi-\zeta\xi\partial_\xi\right],\\
Q_G &= \frac{2}{2N'+2\ell+1}\left(2N'+2\ell+1-z\partial_z-\zeta\partial_\zeta\right)\partial_\xi\,,\\
Q_G^\ddagger &= -\frac{4M-2\ell}{2N'+2\ell+1}\left[\left(2N'+2\ell+1\right)\xi+\zeta\partial_z\right].
\end{aligned} \tag{4.5}$$

The naive hermitian conjugate of a symmetry operator $\mathcal{O}$ will not coincide with its new hermitian conjugate $\mathcal{O}^\ddagger$ unless $\mathcal{O}$ commutes with G. For this reason, it is convenient to choose a basis in which the original SU(2|1) symmetry operators $\mathcal{O}$ are replaced by the operators

$$\tilde{\mathcal{O}} = \mathcal{O} + \frac{1}{2}\mathcal{O}_G G\,, \tag{4.6}$$

which commute with G even when $\mathcal{O}$ does not. This property means that

$$\tilde{\mathcal{O}}^\ddagger = \tilde{\mathcal{O}}^\dagger = \mathcal{O}^\dagger - \frac{1}{2}G\mathcal{O}_G^\ddagger\,. \tag{4.7}$$

In the case that $\mathcal{O}$ is hermitian with respect to the 'naive' Hilbert space metric, the operator $\tilde{\mathcal{O}}$ will be hermitian with respect to the new Hilbert space norm.

When applied to the operators Π and Q, the definition (4.6) yields

$$\begin{aligned}
\tilde\Pi &= \Pi + \frac{1}{2}\Pi_G\,, &\qquad \tilde\Pi^\dagger &= \Pi^\dagger - \frac{1}{2}\Pi_G^\ddagger\,,\\
\tilde Q &= Q + \frac{1}{2}Q_G\,, &\qquad \tilde Q^\dagger &= Q^\dagger - \frac{1}{2}Q_G^\ddagger\,,
\end{aligned} \tag{4.8}$$

where we have used the remarkable identities

$$\Pi_G G = \Pi_G\,, \qquad Q_G G = Q_G\,. \tag{4.9}$$

In terms of the rescaled odd charges

$$\left(\tilde\Pi', \tilde Q'\right) = \sqrt{\frac{2N'+2\ell+1}{2M+2N'+\ell+1}}\left(\tilde\Pi, \tilde Q\right), \tag{4.10}$$

and the redefined U(1) generator

$$F' = F - 2M + \ell\,, \tag{4.11}$$

one finds, after some computation, that the non-zero (anti)commutation relations of the odd charges $(\tilde{\Pi}', \tilde{Q}')$, and their hermitian conjugates, and the even $SU(2) \times U(1)$ charges $(J_3, J_\pm, F')$ are precisely of the standard SU(2|1) form given in section 2.2. Thus, these charges provide an alternative basis for the SU(2|1) symmetry algebra.

Now we turn to the 'hidden' symmetry charges. Their non-zero anticommutators are

$$\begin{aligned}
\left\{\Pi_G, \Pi_G^\ddagger\right\} &= \frac{4\,(\ell-2M)}{2N'+2\ell+1}\left(J_3+\check{F}\right), & \left\{Q_G, Q_G^\ddagger\right\} &= \frac{4\,(\ell-2M)}{2N'+2\ell+1}\left(-J_3+\check{F}\right),\\
\left\{\Pi_G, Q_G^\ddagger\right\} &= -i\frac{4\,(\ell-2M)}{2N'+2\ell+1}\,J_-\,, & \left\{\Pi_G^\ddagger, Q_G\right\} &= i\frac{4\,(\ell-2M)}{2N'+2\ell+1}\,J_+\,,
\end{aligned} \tag{4.12}$$

where

$$\check{F} = 2M + 2N' + \ell + 1 - F\,. \tag{4.13}$$

Notice that the coefficients are level-dependent. The ℓ-dependence in the denominators is easily removed by a level-dependent rescaling of the odd charges but the $(\ell - 2M)$ factor in the numerators is more problematic because when $M = 0$ this factor is zero for $\ell = 0$ but non-zero for $\ell > 0$. For this reason, we will discuss these two cases separately.

4.1 $-2N' - 1 < 2M < 0$

In this case we may define new odd charges by

$$\check{\Pi}_G = -\sqrt{\frac{2N'+2\ell+1}{4\,(\ell-2M)}}\,Q_G^\ddagger\,, \qquad \check{Q}_G = \sqrt{\frac{2N'+2\ell+1}{4\,(\ell-2M)}}\,\Pi_G^\ddagger\,, \tag{4.14}$$

in terms of which the anti-commutation relations of (4.12) become

$$\begin{aligned}
\left\{\check{\Pi}_G, \check{\Pi}_G^\ddagger\right\} &= -J_3 + \check{F}\,, & \left\{\check{Q}_G, \check{Q}_G^\ddagger\right\} &= J_3 + \check{F}\,,\\
\left\{\check{\Pi}_G, \check{Q}_G^\ddagger\right\} &= iJ_-\,, & \left\{\check{\Pi}_G^\ddagger, \check{Q}_G\right\} &= -iJ_+\,.
\end{aligned} \tag{4.15}$$

To present the commutators of these new odd charges with the even charges of SU(2|1) we need give only the non-zero commutators with $(\check{\Pi}_G, \check{Q}_G)$ charges since the remainder are found by hermitian conjugation; these are

$$\begin{aligned}
\left[\check{F}, \check{\Pi}_G\right] &= -\frac{1}{2}\check{\Pi}_G\,, & \left[\check{F}, \check{Q}_G\right] &= -\frac{1}{2}\check{Q}_G\,,\\
\left[J_3, \check{\Pi}_G\right] &= -\frac{1}{2}\check{\Pi}_G\,, & \left[J_3, \check{Q}_G\right] &= \frac{1}{2}\check{Q}_G\,,\\
\left[J_+, \check{\Pi}_G\right] &= i\check{Q}_G\,, & \left[J_-, \check{Q}_G\right] &= -i\check{\Pi}_G\,.
\end{aligned} \tag{4.16}$$

This shows that the new odd symmetry charges transform as a charged doublet under the U(2) subgroup of SU(2|1). In fact, the operators $(\check{\Pi}^\ddagger, \check{Q}^\ddagger)$, together with their hermitian conjugates, and the even charges $(J_3, J_\pm, \check{F})$, obey the (anti)commutation relations of SU(2|1) given in (2.2). The full symmetry group therefore contains two distinct SU(2|1) superalgebras. As F' is the U(1) charge of one of these superalgebras and $\check{F}$ the U(1) charge of the other one, the full symmetry group must contain

$$Z = F' + \check{F} = 2N' + 2\ell + 1\,, \tag{4.17}$$

which is a level-dependent central charge. However, this level-dependence does not present a problem; it just means that we have a central charge

$$Z = 2L + 2N' + 1\,, \tag{4.18}$$

where L is the level operator .

The two SU(2|1) superalgebras are non-commuting because there are non-zero anti-commutators of the odd charges from one with the odd charges from the other. These are

$$\begin{aligned} \left\{\tilde{\Pi}', \check{\Pi}_G^{\ddagger}\right\} &= \left\{\tilde{Q}', \check{Q}_G^{\ddagger}\right\} = i\mathcal{J}_-\,, \\ \left\{\tilde{\Pi}'^{\ddagger}, \check{\Pi}_G\right\} &= \left\{\tilde{Q}'^{\ddagger}, \check{Q}_G\right\} = -i\mathcal{J}_+\,, \end{aligned} \tag{4.19}$$

where the analytic operators representing $\mathcal{J}_\pm$ are

$$\begin{aligned} \mathcal{J}_+ &= i\sqrt{(\ell - 2M)\,(2M + 2N' + \ell + 1)}\,\xi\zeta\,, \\ \mathcal{J}_- &= \frac{i}{\sqrt{(\ell - 2M)\,(2M + 2N' + \ell + 1)}}\,\partial_\xi\partial_\zeta\,. \end{aligned} \tag{4.20}$$

These satisfy, together with

$$\mathcal{J}_3 = \frac{1}{2}\left(-1 + \xi\partial_\xi + \zeta\partial_\zeta\right), \tag{4.21}$$

the standard $su(2)$ commutation relations

$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_3\,, \qquad [\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm\,. \tag{4.22}$$

Finally, the non-zero commutators of these new SU(2) charges with the odd charges are

$$\begin{aligned} \left[\mathcal{J}_+, \tilde{\Pi}'\right] &= -i\check{\Pi}_G, & \left[\mathcal{J}_+, \tilde{Q}'\right] &= i\check{Q}_G\,, \\ \left[\mathcal{J}_-, \Pi'_G\right] &= i\tilde{\Pi}', & \left[\mathcal{J}_-, \check{Q}_G\right] &= -i\tilde{Q}'\,, \\ \left[\mathcal{J}_3, \tilde{\Pi}'\right] &= -\frac{1}{2}\tilde{\Pi}'\,, & \left[\mathcal{J}_3, \tilde{Q}'\right] &= -\frac{1}{2}\tilde{Q}'\,, \\ \left[\mathcal{J}_3, \check{\Pi}_G\right] &= \frac{1}{2}\check{\Pi}_G\,, & \left[\mathcal{J}_3, \check{Q}_G\right] &= \frac{1}{2}\check{Q}_G\,, \end{aligned} \tag{4.23}$$

and hermitian conjugates. These commutation relations show that $(\tilde{\Pi}', \check{\Pi}_G)$ and $(\tilde{Q}', \check{Q}_G)$ are doublets of the SU(2) group generated by $(\mathcal{J}_\pm, \mathcal{J}_3)$.

We have now shown that the charges

$$\{J_\pm, J_3, \mathcal{J}_\pm, \mathcal{J}_3, Z; \tilde{\Pi}', \tilde{Q}'; \check{\Pi}_G, \check{Q}_G\} \tag{4.24}$$

span a Lie superalgebra, with structure constants that are level independent. We have therefore found a finite-dimensional 'enlarged' symmetry algebra. The brackets where the central charge Z defined in (4.18) contributes, are:

$$\begin{aligned} \{\check{\Pi}_G, \check{\Pi}_G^{\ddagger}\} &= -J_3 - \mathcal{J}_3 + Z\,, & \{\check{Q}_G, \check{Q}_G^{\ddagger}\} &= J_3 - \mathcal{J}_3 + Z\,, \\ \{\tilde{\Pi}', \tilde{\Pi}'^{\dagger}\} &= -J_3 + \mathcal{J}_3 + Z\,, & \{\tilde{Q}', \tilde{Q}'^{\dagger}\} &= J_3 + \mathcal{J}_3 + Z\,. \end{aligned} \tag{4.25}$$

Its even subalgebra is that of SU(2)×SU(2)×U(1), where the U(1) charge is central, and its four complex odd generators transform as the $(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})$ of SU(2)×SU(2). This uniquely fixes the full symmetry algebra to be that of SU(2|2); recall that the groups SU($p|q$) have even subgroup SU(p) × SU(q) × U(1) with the U(1) charge being central when $p = q$.

4.1.1 Casimir considerations

Acting on the wave functions at the ℓth level, the SU(2|1) Casimir operators (2.5) and (2.6) for the superflag model become

$$C_2 = (\ell - 2M)\left(2M + 2N' + \ell + 1\right), \qquad C_3 = \left(4M + 2N' + 1\right) C_2\,. \tag{4.26}$$

For the general superflag model, one has

$$H = C_2 + 2M\left(2M + 2N' + 1\right). \tag{4.27}$$

At levels for which $C_2 = 0$, which is possible when $2M$ is a non-negative integer, then $C_3 = 0$ too, and hence the SU(2|1) representation is 'atypical'. In particular, $C_2 = C_3 = 0$ for the LLL when $M = 0$, in which case

$$H = C_2\big|_{M=0} = \ell\left(\ell + 2N' + 1\right), \tag{4.28}$$

in agreement with our result of (2.33) for the supersphere if we make the identification

$$2N' = 2N - 1\,. \tag{4.29}$$

The $su(2|2)$ symmetry algebra for $M < 0$ is a subalgebra of the enveloping algebra of $su(2|1)$. To see this we define the following functions of the Casimir operators:

$$\begin{aligned} \mathcal{A} &= \sqrt{\frac{C_3 + \sqrt{C_3^2 + 4C_2^3}}{2C_2\,\sqrt{C_3^2 + 4C_2^3}}}\,, \\ \mathcal{B} &= \sqrt{\frac{1}{2C_2^3}\left(C_3^2 + 4C_2^3 - C_3\sqrt{C_3^2 + 4C_2^3}\right)}\,, \\ \mathcal{C} &= \sqrt{\frac{1}{2C_2^3}\left(C_3^2 + 2C_2^3 - C_3\sqrt{C_3^2 + 4C_2^3}\right)}\,. \end{aligned} \tag{4.30}$$

The odd charges of SU(2|2) may now be written as

$$\begin{aligned} \check{\Pi}_G &= \mathcal{A}\left[iJ_-\Pi^\dagger + Q^\dagger\left(J_3 - F + \frac{C_3}{2C_2} - \sqrt{\left(\frac{C_3}{2C_2}\right)^2 + C_2}\right)\right], \\ \check{Q}_G &= \mathcal{A}\left[iQ^\dagger J_+ + \Pi^\dagger\left(J_3 + F + 1 - \frac{C_3}{2C_2} + \sqrt{\left(\frac{C_3}{2C_2}\right)^2 + C_2}\right)\right], \\ \tilde{\Pi}' &= \mathcal{B}\Pi + \mathcal{C}\check{Q}_G^\ddagger\,, \qquad \tilde{Q}' = \mathcal{B}Q - \mathcal{C}\check{\Pi}_G^\ddagger\,. \end{aligned} \tag{4.31}$$

The even charges are those of the original SU(2) symmetry, $(J_\pm, J_3)$, the central charge $Z = L + 2N + 1$, and the 'hidden' SU(2) charges

$$\mathcal{J}_- = \frac{i}{\sqrt{C_2}}\,\Pi Q\,, \qquad \mathcal{J}_+ = i\sqrt{C_2}\,Q^\dagger\Pi^\dagger, \qquad \mathcal{J}_3 = F - \frac{C_3}{2C_2}\,. \tag{4.32}$$

4.2 $M = 0$ and the planar limit

In this case the anticommutation relations (4.12) reduce to

$$\begin{aligned}
\{\Pi_G, \Pi_G^\ddagger\} &= \frac{4\ell}{2N+2\ell}\left(J_3 + \check{F}\right), \\
\{Q_G, Q_G^\ddagger\} &= \frac{4\ell}{2N+2\ell}\left(-J_3 + \check{F}\right), \\
\{\Pi_G, Q_G^\ddagger\} &= -i\frac{4\ell}{2N+2\ell}J_-\,,
\end{aligned} \tag{4.33}$$

where we have used (4.29). As all these anti-commutators vanish for $\ell = 0$, the LLL states must be annihilated by *both* (Π_G, Q_G) and their hermitian conjugates $(\Pi_G^\ddagger, Q_G^\ddagger)$. At higher levels, we get supermultiplets of states that may be constructed by the repeated action of $(\Pi_G^\ddagger, Q_G^\ddagger)$ on 'Clifford vacuum' states annihilated by (Π_G, Q_G). In fact, all higher levels may be shown to form representations of SU(2|2) by the argument just used to analyze all levels when $M < 0$. However, because of the exceptional LLL for $M = 0$, one cannot say that the model has an SU(2|2) symmetry. Neither is there a conventional supersymmetry, as there is in the planar limit, because the commutators of the 'supersymmetry' generators (Π_G, Q_G) with the even generators of SU(2|1) produce further odd symmetry generators. In fact, closure of the algebra appears to require an infinite number of generators.

As this state of affairs is in marked contrast to the simple results obtained in [4] for the superplane Landau model, we now discuss how those results may be recovered in the planar limit. To do so we must restore dependence on the radius R of the sphere that is the 'body' of both the supersphere and superflag supermanifolds. Specifically, the Hamiltonian must be rescaled:

$$H \to H/R^2 = 2\ell\left(N/R^2 + \ell/R^2\right)\,. \tag{4.34}$$

We then take $R \to \infty$, keeping fixed

$$\kappa = N/R^2\,. \tag{4.35}$$

This gives

$$H_{\text{superplane}} = 2\kappa L\,, \tag{4.36}$$

where L is the level operator with eigenvalue ℓ on the ℓ th level. This agrees with [3, 4] after taking into account the difference in notations of that paper.[7]

From the N dependence of the generators $(J_\pm, J_3, \check{F})$ we find that

$$\begin{aligned}
J_-/R^2 &= \mathcal{O}\left(1/R^2\right), & J_+ &= -2i\kappa z + \mathcal{O}\left(1/R^2\right), \\
\check{F} + J_3 &= \mathcal{O}\left(1/R^2\right), & \check{F} - J_3 &= 2\kappa + \mathcal{O}\left(1/R^2\right).
\end{aligned} \tag{4.37}$$

The anti-commutation relations (4.33) can now be written as

$$\{\Pi_G, \Pi_G^\ddagger\} = \mathcal{O}\left(1/R^2\right), \qquad \{\Pi_G, Q_G^\ddagger\} = \mathcal{O}\left(1/R^2\right), \tag{4.38}$$

[7] Confusingly for present purposes, the level number ℓ was called N in [3, 4]. The parameter N used here does not appear as such in the planar limit because it is replaced by the real number κ.

and

$$\{Q_G, Q_G^\ddagger\} = 2\ell + \mathcal{O}\left(1/R^2\right). \tag{4.39}$$

Thus, only Q_G survives the planar limit, and it is proportional to the worldline supersymmetry charge S of [4].

5. Supersphere from superflag

In this section we show how the quantum states of the supersphere model and its Hamiltonian can be recovered using the basic geometric objects of the superflag manifold $\mathrm{SU}(2|1)/[\mathrm{U}(1)\times\mathrm{U}(1)]$. The supersphere $\mathrm{SU}(2|1)/\,\mathrm{U}(1|1)$ is an $\mathrm{SU}(2|1)$ invariant subspace in $\mathrm{SU}(2|1)/[\mathrm{U}(1)\times\mathrm{U}(1)]$, whence it follows that any considerations related to the supersphere should have an equivalent formulation in terms of the properly constrained objects defined on the superflag. Throughout this section we assume that all wave functions are superfields, i.e. that they have definite Grassmann parity.

5.1 Covariant derivatives

As shown in [2], the geometry of the superflag manifold $\mathrm{SU}(2|1)/[\mathrm{U}(1)\times\mathrm{U}(1)]$ is described by a set of covariant derivatives with non-trivial $\mathrm{U}(1)\times\mathrm{U}(1)$ connections:

$$\begin{aligned}
&\mathcal{D}^- = D^- - (D^-\log K_2)\hat J_3 + (D^-\log K_1)\hat B\,, \qquad \mathcal{D}^+ = D^+ + (D^+\log K_1)\hat B \\
&\bar{\mathcal{D}}^- = \bar D^- - (\bar D^-\log K_1)\hat B\,, \qquad \bar{\mathcal{D}}^+ = \bar D^+ + (\bar D^+\log K_2)\hat J_3 - (\bar D^+\log K_1)\hat B\,, \\
&\mathcal{D}^{--} = D^{--} - (D^{--}\log K_2)\,\hat J_3\,, \qquad \mathcal{D}^{++} = D^{++} + (D^{++}\log K_2)\,\hat J_3\,,
\end{aligned} \tag{5.1}$$

subject to the conjugation rules[8]

$$\bar{\mathcal{D}}^+ = \overline{(\mathcal{D}^-)}\,, \qquad \bar{\mathcal{D}}^- = \overline{(\mathcal{D}^+)}\,, \qquad \mathcal{D}^{++} = \overline{(\mathcal{D}^{--})}\,. \tag{5.2}$$

Explicit expressions for the covariant derivatives were given in [2] for local superflag coordinates (z,ξ^1,ξ^2), where $(\xi^1,\xi^2) = (\zeta + z\xi, \xi)$. Here we use the local coordinates (z,ζ,ξ), in which case

$$\begin{aligned}
D^- &= (K_1K_2)^{\frac12}\left\{\partial_\zeta - K_2^{-1}\left(\bar z - \xi\bar\zeta\right)\partial_\xi + K_1^{-1}\left[(1+z\bar z)\bar\xi + z\bar\zeta\right](\partial_z - \xi\partial_\zeta)\right\},\\
\bar D^+ &= -(K_1K_2)^{\frac12}\left\{\partial_{\bar\zeta} - K_2^{-1}\left(z + \bar\xi\zeta\right)\partial_{\bar\xi} - K_1^{-1}\left[(1+z\bar z)\xi + \bar z\zeta\right]\left(\partial_{\bar z} - \bar\xi\partial_{\bar\zeta}\right)\right\},\\
D^+ &= K_2^{-\frac12}\partial_\xi\,, \qquad \bar D^- = -K_2^{-\frac12}\partial_{\bar\xi}\,,\\
D^{--} &= K_1^{\frac12}K_2\left(\partial_z - \xi\partial_\zeta\right), \quad D^{++} = K_1^{\frac12}K_2\left(\partial_{\bar z} - \bar\xi\partial_{\bar\zeta}\right).
\end{aligned} \tag{5.3}$$

The functions K_1 and K_2 are given for our choice of coordinates[9] in (3.4). In (5.1), the operators $\hat B$, $\hat J_3$ are 'matrix' parts of the $\mathrm{U}(1)$ generators J_3 and B, where B is related to the generator F of the previous sections by

$$B = \frac12\,(F - J_3)\,. \tag{5.4}$$

[8]These rules are the same as those for the purely 'derivative' parts of the covariant derivatives.

[9]The expressions in [2] differ because of the different coordinates used there.

The covariant derivatives have the following commutation relations with the operator $\hat{F}$:

$$\left[\hat{F}\,,\mathcal{D}^{\pm}\right]=\frac{1}{2}\,\mathcal{D}^{\pm}\,,\qquad \left[\hat{F}\,,\bar{\mathcal{D}}^{\pm}\right]=-\frac{1}{2}\,\bar{\mathcal{D}}^{\pm}\,,\qquad \left[\hat{F}\,,\mathcal{D}^{\pm\pm}\right]=0\,. \tag{5.5}$$

It is also useful to have the commutation relations with the operators $\hat{J}_3$ and $\hat{B}$:

$$\begin{aligned}
&[\hat{J}_3\,,\mathcal{D}^{\pm}]=\pm\frac{1}{2}\,\mathcal{D}^{\pm}\,, && [\hat{J}_3\,,\bar{\mathcal{D}}^{\pm}]=\pm\frac{1}{2}\,\bar{\mathcal{D}}^{\pm}\,, && [\hat{J}_3\,,\mathcal{D}^{\pm\pm}]=\pm\mathcal{D}^{\pm\pm}\,,\\
&[\hat{B}\,,\mathcal{D}^{\pm\pm}]=\mp\frac{1}{2}\,\mathcal{D}^{\pm\pm}\,, && [\hat{B}\,,\mathcal{D}^{-}]=\frac{1}{2}\,\mathcal{D}^{-}\,, && [\hat{B}\,,\bar{\mathcal{D}}^{+}]=-\frac{1}{2}\,\bar{\mathcal{D}}^{+}\,,\\
&[\hat{B}\,,\mathcal{D}^{+}]=[\hat{B}\,,\bar{\mathcal{D}}^{-}]=0\,.
\end{aligned} \tag{5.6}$$

In what follows, a crucial role will be played by the (anti)commutation relations between the covariant derivatives:

$$\begin{aligned}
&\{\mathcal{D}^{-},\bar{\mathcal{D}}^{-}\}=-\mathcal{D}^{--}\,, && \{\mathcal{D}^{+},\bar{\mathcal{D}}^{+}\}=\mathcal{D}^{++}\,,\\
&\{\mathcal{D}^{-},\bar{\mathcal{D}}^{+}\}=2(\hat{B}+\hat{J}_3)=\hat{F}+\hat{J}_3\,, && \{\mathcal{D}^{+},\bar{\mathcal{D}}^{-}\}=2\hat{B}=\hat{F}-\hat{J}_3\,,\\
&\{\mathcal{D}^{-},\mathcal{D}^{+}\}=\{\bar{\mathcal{D}}^{-},\bar{\mathcal{D}}^{+}\}=\{\mathcal{D}^{\pm},\mathcal{D}^{\pm}\}=\{\bar{\mathcal{D}}^{\pm},\bar{\mathcal{D}}^{\pm}\}=0\,,\\
&[\mathcal{D}^{++},\mathcal{D}^{-}]=-\mathcal{D}^{+},\ [\mathcal{D}^{++},\mathcal{D}^{+}]=0\,, && [\mathcal{D}^{++},\bar{\mathcal{D}}^{-}]=\bar{\mathcal{D}}^{+},\ [\mathcal{D}^{++},\bar{\mathcal{D}}^{+}]=0\,,\\
&[\mathcal{D}^{--},\mathcal{D}^{+}]=\mathcal{D}^{-},\quad [\mathcal{D}^{--},\mathcal{D}^{-}]=0\,, && [\mathcal{D}^{--},\bar{\mathcal{D}}^{+}]=-\bar{\mathcal{D}}^{-},\ [\mathcal{D}^{--},\bar{\mathcal{D}}^{-}]=0\,,\\
&[\mathcal{D}^{++},\mathcal{D}^{--}]=-2\,\hat{J}_3\,.
\end{aligned} \tag{5.7}$$

These relations are equivalent to the Maurer-Cartan equations for the left-invariant 1-forms on the superflag SU(2|1)/[U(1)×U(1)], and so fully encode the geometry of this supercoset manifold. They can be derived from the Maurer-Cartan equations on the superflag manifold without reference to the explicit form of the covariant derivatives.

5.1.1 Superflag superfields

The U(1) × U(1) operators $(\hat{J}_3, \hat{B})$ have eigenvalues $(\hat{N}, \hat{M})$. Let $\Psi^{(\hat{N},\hat{M})}$ denote an eigenfunction of these operators:

$$\hat{J}_3\,\Psi^{(\hat{N},\hat{M})}=\hat{N}\,\Psi^{(\hat{N},\hat{M})}\,,\qquad \hat{B}\,\Psi^{(\hat{N},\hat{M})}=\hat{M}\,\Psi^{(\hat{N},\hat{M})}\,. \tag{5.8}$$

A covariant derivative of any such eigenfunction (which is a superfield on the superflag manifold) is another eigenfunction because the covariant derivatives have definite U(1) × U(1) charges as a consequence of the commutation relations (5.5) and (5.6).

The general SU(2|1)/[U(1)×U(1)] superfields $\Psi^{(\hat{N},\hat{M})}$ have the following transformation law under the odd SU(2|1) transformations [2]:

$$\begin{aligned}
\delta\Psi^{(\hat{N},\hat{M})}\ =\ &-\hat{N}\left(\epsilon^1\bar{\zeta}+\bar{\epsilon}_1\zeta\right)\Psi^{(\hat{N},\hat{M})}\\
&-\hat{M}\left[\epsilon^1(\bar{\zeta}+\bar{z}\bar{\xi})+\epsilon^2\bar{\xi}+\bar{\epsilon}_1(\zeta+z\xi)+\bar{\epsilon}_2\xi\right]\Psi^{(\hat{N},\hat{M})}\,.
\end{aligned} \tag{5.9}$$

It should be appreciated that the SU(2|1)/[U(1) × U(1)] superfields defined by (5.8) are purely geometric objects having no *a priori* relation to the quantum superflag or supersphere wave superfunctions that we discussed in the previous sections. Consequently, the real eigenvalues $(\hat{N}, \hat{M})$ are not obliged to coincide with the model parameters N, N' and M appearing in the Lagrangians (2.20) and (3.8). Nevertheless, it will turn out that the wave functions of the quantum superflag Landau model are superfields $\tilde{\Psi}^{(\hat{N},\hat{M})}$ with $\hat{N} = N', \hat{M} = M$ that satisfy, in addition to the general $\mathrm{U}(1) \times \mathrm{U}(1)$ stability subgroup conditions (5.8), the chirality constraints

$$\bar{\mathcal{D}}^{+}\tilde{\Psi}^{(\hat{N},\hat{M})} = \bar{\mathcal{D}}^{-}\tilde{\Psi}^{(\hat{N},\hat{M})} = 0\,. \tag{5.10}$$

5.1.2 Supersphere superfields

Let us consider another particular class of the general superfields $\Psi^{(\hat{N},\hat{M})}$ defined by (5.8), namely those subject to the restriction

$$\mathcal{D}^{+}\Psi^{(\hat{N},\hat{M})} = \bar{\mathcal{D}}^{-}\Psi^{(\hat{N},\hat{M})} = 0\,. \tag{5.11}$$

By virtue of the second anticommutation relation in (5.7), these constraints are compatible with a non-zero superfield only when $\hat{M} = 0$. Then, from the definition (5.1), it follows that $\mathcal{D}^{+} = D^{+}$ and $\bar{\mathcal{D}}^{-} = \bar{D}^{-}$ when these operators act on the $\Psi^{(\hat{N},0)}$ superfields. Recalling the precise expressions for D^{+} and $\bar{D}^{-}$ from (5.3), we conclude that (5.11) is equivalent, for $\hat{M} = 0$, to

$$\partial_{\xi}\Psi^{(\hat{N},0)} = \partial_{\bar{\xi}}\Psi^{(\hat{N},0)} = 0\,. \tag{5.12}$$

In other words, for $\hat{M} = 0$ the general $\mathrm{SU}(2|1)/[\mathrm{U}(1)\times\mathrm{U}(1)]$ superfields may be consistently restricted, by the covariant conditions (5.11), to supersphere superfields, which have no dependence on the Grassmann-odd complex coordinate ξ and its complex conjugate $\bar{\xi}$.

As we shall see soon, the wave superfunctions of the quantum supersphere model indexed by N belong to this subclass of the $\mathrm{SU}(2|1)/[\mathrm{U}(1) \times \mathrm{U}(1)]$ superfields in which one should identify $\hat{N} = N$.

5.1.3 Casimir operators

For what follows, it will be instructive to rewrite the quadratic and cubic SU(2|1) Casimir operators (2.5) and (2.6) in terms of the above covariant derivatives. These are

$$C_2 = \frac{1}{2}\left(2(\hat{J}_3)^2 - \{\mathcal{D}^{++}, \mathcal{D}^{--}\} - [\bar{\mathcal{D}}^{+}, \mathcal{D}^{-}] - [\bar{\mathcal{D}}^{-}, \mathcal{D}^{+}]\right) - (\hat{F})^2\,, \tag{5.13}$$

$$\begin{aligned} C_3 = {} & \frac{1}{4}\left(\{\mathcal{D}^{--}, [\mathcal{D}^{+}, \bar{\mathcal{D}}^{+}]\} - \{\mathcal{D}^{++}, [\mathcal{D}^{-}, \bar{\mathcal{D}}^{-}]\}\right) \\ & + \frac{1}{4}\{\hat{J}_3, [\mathcal{D}^{+}, \bar{\mathcal{D}}^{-}] - [\mathcal{D}^{-}, \bar{\mathcal{D}}^{+}]\} - \frac{1}{2}\{\hat{F}, \{\mathcal{D}^{++}, \mathcal{D}^{--}\} - 2(\hat{J}_3)^2\} \\ & + \frac{3}{4}\{\hat{F}, [\mathcal{D}^{+}, \bar{\mathcal{D}}^{-}] + [\mathcal{D}^{-}, \bar{\mathcal{D}}^{+}]\} - 2(\hat{F})^3 - \hat{F}\,. \end{aligned} \tag{5.14}$$

Perhaps, the simplest way to prove the coincidence of (5.13) and (5.14) with (2.5) and (2.6) on general superflag superfields is to use one more equivalent expression of the same invariant operators through the SU(2|1) generators in the manifestly $\mathrm{U}(2)$ covariant basis:

$$C_2 = \frac{1}{2}[\bar{Q}^k, Q_k] - F^2 - \frac{1}{2}T^{ik}T_{ik}\,, \tag{5.15}$$

$$C_3 = \frac{1}{4}\{T^{ik}, [Q_i, \bar{Q}_k]\} + \frac{3}{4}\{F, [Q^i, \bar{Q}_i]\} - \frac{1}{2}\{F, T^{ik}T_{ik}\} - 2F^3 - F\,. \tag{5.16}$$

In this basis, the (anti)commutation relations of the superalgebra $su(2|1)$ are

$$
\begin{aligned}
\{Q_i, \bar{Q}_k\} &= \epsilon_{ik} F + T_{ik}\,, & \{Q_i, Q_k\} &= \{\bar{Q}_i, \bar{Q}_k\} = 0\,,\\
[T_{ik}, Q_l] &= \frac{1}{2}\left(\epsilon_{il} Q_k + \epsilon_{kl} Q_i\right), & [T_{ik}, \bar{Q}_l] &= \frac{1}{2}\left(\epsilon_{il} \bar{Q}_k + \epsilon_{kl} \bar{Q}_i\right),\\
[F, Q_l] &= \frac{1}{2}\, Q_l\,, & [F, \bar{Q}_l] &= -\frac{1}{2}\, \bar{Q}_l\,,\\
[T_{ik}, T_{lj}] &= \epsilon_{ij} T_{kl} + \epsilon_{kl} T_{ij}\,, & \bar{Q}^i &= (Q_i)^\dagger\,.
\end{aligned} \tag{5.17}
$$

5.2 Supersphere in terms of the superflag superfields

As was noticed in section 2.3 (eq. (2.31)), the supersphere Hamiltonian (2.27) coincides with the Casimir operator C_2. On the other hand, in section 2.4, based on considering invariant norms, it was anticipated that the quantum supersphere model at $2N$ is equivalent to the particular case of the quantum superflag model at $2N' = 2N - 1$ and $M = 0$. Following this observation, we are led to consider the operator (5.13) at $\hat{M} = M = 0$, i.e. at

$$(\hat{F})^2 - (\hat{J}_3)^2 = 0\,, \tag{5.18}$$

as the appropriate 'would-be' Hamiltonian of the supersphere in the manifestly covariant formulation through superflag superfields

$$H = -\frac{1}{2}\left(\mathcal{D}^{++}\mathcal{D}^{--} + \mathcal{D}^{--}\mathcal{D}^{++}\right) + \frac{1}{2}[\mathcal{D}^-, \bar{\mathcal{D}}^+] + \frac{1}{2}[\mathcal{D}^+, \bar{\mathcal{D}}^-]\,. \tag{5.19}$$

In section 5.4 we shall prove that this operator indeed reduces to (2.27) on the properly constrained SU(2|1)/[U(1)×U(1)] superfields. Moreover, being restricted to the general superflag model wave functions (5.10) (with $\hat{M} = M \neq 0$), it reduces to the covariant form of the superflag model Hamiltonian (3.22) (modulo a constant shift, see eq. (5.24) below). Thus it can be regarded as a sort of 'master' Hamiltonian for these two different quantum models.

Now we present the covariant form of the supersphere wave functions satisfying (5.11). The supersphere LLL wave function $\Psi_0^{(N,0)}$ is defined by the conditions

$$(\mathrm{a}):\ \mathcal{D}^+\Psi_0^{(N,0)} = \bar{\mathcal{D}}^-\Psi_0^{(N,0)} = 0\,; \qquad (\mathrm{b}):\ \bar{\mathcal{D}}^+\Psi_0^{(N,0)} = \mathcal{D}^{++}\Psi_0^{(N,0)} = 0\,. \tag{5.20}$$

It can be easily checked that such functions are annihilated by the operator H of (5.19) as a consequence of the (anti)commutation relations (5.7):

$$H\,\Psi_0^{(N,0)} = 0\,. \tag{5.21}$$

The conditions (5.20a) put $\Psi_0^{(N,0)}$ on the supersphere, eliminating the dependence on $(\xi, \bar{\xi})$. Then eqs. (5.20b) are the covariant chirality conditions which effectively eliminate the dependence on $\bar{z}$ and $\bar{\zeta}$; this can be made manifest by solving these constraints.

Already on this simplest example one can explicitly see the equivalence relation between the $M = 0$ superflag model and the supersphere model anticipated in the previous sections. The supersphere LLL wave function can be represented as

$$\Psi_0^{(N,0)} = \mathcal{D}^+\tilde{\Psi}_0^{(N-\frac{1}{2},0)}\,, \tag{5.22}$$

where $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$ obeys the constraints

$$\bar{\mathcal{D}}^- \tilde{\Psi}_0^{(N-\frac{1}{2},0)} = \bar{\mathcal{D}}^+ \tilde{\Psi}_0^{(N-\frac{1}{2},0)} = \mathcal{D}^{++} \tilde{\Psi}_0^{(N-\frac{1}{2},0)} = 0\,. \tag{5.23}$$

The wave function defined by (5.22) satisfies the constraints (5.20) as a consequence of (5.23) and the (anti)commutation relations (5.7) at $M = 0$ (in particular, the relation $\mathcal{D}^+\mathcal{D}^+ = 0$). Now let us examine the superfunction $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$. It is covariantly chiral and analytic. We should also take into account that on the general set of superfunctions $\tilde{\Psi}^{(\hat{N},\hat{M})}$ obeying the chirality conditions (5.10) the operator (5.19) is reduced, modulo a constant shift by $2\hat{M}$, to the superflag Hamiltonian in the covariant formulation [2]

$$H \quad \Rightarrow \quad H'_{SF} = -\mathcal{D}^{--}\mathcal{D}^{++} - 2\hat{M} = H_{SF} - 2\hat{M}\,. \tag{5.24}$$

The superfunction $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$ is a particular case of these general chiral functions corresponding to $\hat{N} = N' = N - \frac{1}{2}\,, \hat{M} = M = 0$ and satisfying the additional analyticity condition $\mathcal{D}^{++}\tilde{\Psi}_0^{(N-\frac{1}{2},0)} = 0$. Hence, $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$ is just the LLL wave function for the $N' = N - \frac{1}{2}$ superflag model at $\hat{M} = M = 0$. It should be pointed out that $\Psi_0^{(N,0)}$ is Grassmann-odd if $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$ is Grassmann-even and vice versa. The first option precisely matches with our previous choice of the Grassmann parity of the wave superfunctions in the supersphere and superflag models. Note that (5.22) admits gauge invariance

$$\begin{aligned} \tilde{\Psi}_0^{(N-\frac{1}{2},0)\,\prime} &= \tilde{\Psi}_0^{(N-\frac{1}{2},0)} + \Lambda^{(N-\frac{1}{2},0)}\,, \\ \mathcal{D}^+\Lambda^{(N-\frac{1}{2},0)} &= \bar{\mathcal{D}}^+\Lambda^{(N-\frac{1}{2},0)} = \bar{\mathcal{D}}^-\Lambda^{(N-\frac{1}{2},0)} = \mathcal{D}^{++}\Lambda^{(N-\frac{1}{2},0)} = 0\,. \end{aligned} \tag{5.25}$$

This can be used to remove half the component fields from $\tilde{\Psi}_0^{(N-\frac{1}{2},0)}$ and to equate the numbers of the independent component fields in the left-hand and right-hand sides of (5.22); this number is just $(2+2)$, i.e. that of the 'ultrashort' multiplet of SU(2|1).

To single out, in the variety of the SU(2|1)/[U(1) × U(1)] superfields, the supersphere wave functions related to an $\ell > 0$ level, we will proceed in the following two-step way. First, we define the Grassmann-odd function

$$\Psi_\ell^{(N,0)} = (\mathcal{D}^{--})^\ell \Phi^{(N+\ell,-\frac{\ell}{2})}\,, \tag{5.26}$$

where the relevant 'ground state wave function' $\Phi^{(N+\ell,-\frac{\ell}{2})}$ is also fermionic and satisfies the conditions

$$(\mathrm{a}):\ \mathcal{D}^+\Phi^{(N+\ell,-\frac{\ell}{2})} = 0\,; \qquad (\mathrm{b}):\ \bar{\mathcal{D}}^+\Phi^{(N+\ell,-\frac{\ell}{2})} = \mathcal{D}^{++}\Phi^{(N+\ell,-\frac{\ell}{2})} = 0\,. \tag{5.27}$$

It is straightforward to check that

$$H\,\Psi_\ell^{(N,0)} = (2N\ell + \ell^2)\,\Psi_\ell^{(N,0)}\,, \tag{5.28}$$

as a consequence of (5.7) and (5.27). Thus, $\Psi_\ell^{(N,0)}$ is an eigenfunction of the operator (5.19) with the same eigenvalue as in (2.33). Note that the condition (5.27a) still leaves the dependence on $\bar{\xi}$ in $\Phi^{(N+\ell,-\frac{\ell}{2})}$.

Another set of eigenfunctions of the 'would-be' Hamiltonian (5.19) is

$$\widehat{\Psi}_\ell^{(N,0)} = \mathcal{D}^-(\mathcal{D}^{--})^{\ell-1}\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\,, \tag{5.29}$$

where

$$\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = \bar{\mathcal{D}}^-\Phi^{(N+\ell,-\frac{\ell}{2})}\,. \tag{5.30}$$

It is easy to check that

$$H\,\widehat{\Psi}_\ell^{(N,0)} = (2N\ell+\ell^2)\,\widehat{\Psi}_\ell^{(N,0)}\,. \tag{5.31}$$

The bosonic reduced wave function $\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$ satisfies the conditions

$$(\mathrm{a}):\ \bar{\mathcal{D}}^-\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = 0\,, \qquad (\mathrm{b}):\ \bar{\mathcal{D}}^+\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = \mathcal{D}^{++}\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = 0\,, \tag{5.32}$$

which follow from (5.27) on taking into account the relations (5.7). Using the relation (5.30), eq.(5.29) can be rewritten as

$$\widehat{\Psi}_\ell^{(N,0)} = \mathcal{D}^-\bar{\mathcal{D}}^-(\mathcal{D}^{--})^{\ell-1}\Phi^{(N+\ell,-\frac{\ell}{2})}\,. \tag{5.33}$$

In other words, both series of eigenfunctions can be produced from the single fermionic 'ground state' wave function $\Phi^{(N+\ell,-\frac{\ell}{2})}$ by applying to it the operators $\mathcal{D}^{--}$ and $\mathcal{D}^-\bar{\mathcal{D}}^-$. The latter operator can appear only once, as in (5.33), due to the reduction relation

$$(\mathcal{D}^-\bar{\mathcal{D}}^-)\,(\mathcal{D}^-\bar{\mathcal{D}}^-) = -(\mathcal{D}^-\bar{\mathcal{D}}^-)\,\mathcal{D}^{--}\,, \tag{5.34}$$

which follows from (5.7). Actually, $\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$ is just the covariant definition of the highest component in the $\bar{\xi}$-expansion of $\Phi^{(N+\ell,-\frac{\ell}{2})}$.

The next step consists in representing the 'reduced wave function' $\Phi^{(N+\ell,-\frac{\ell}{2})}$ in (5.26) as

$$\Phi^{(N+\ell,-\frac{\ell}{2})} = \mathcal{D}^+\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\,. \tag{5.35}$$

The 'prepotential' $\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$ is assumed to satisfy the conditions

$$\bar{\mathcal{D}}^-\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = \bar{\mathcal{D}}^+\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = \mathcal{D}^{++}\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} = 0\,, \tag{5.36}$$

and hence can be identified with the level ℓ reduced wave function of the superflag model with the U(1) charges $2N' = 2N-1, M = 0$. The corresponding full wave functions are defined by

$$\tilde{\Psi}_\ell^{(N-\frac{1}{2},0)} = (\mathcal{D}^{--})^\ell\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\,, \tag{5.37}$$

and on them the operator H is reduced (cf. (5.24)) to

$$H \;\Rightarrow\; H_{SF(M=0)} = -\mathcal{D}^{--}\mathcal{D}^{++}\,, \tag{5.38}$$

with the eigenvalues

$$E_\ell = (2N-1)\ell + \ell(\ell+1)\,. \tag{5.39}$$

The constraints (5.27) are satisfied as a consequence of (5.36) and the relation $\mathcal{D}^+\mathcal{D}^+ = 0$. There emerge no additional constraints on $\Phi^{(N+\ell,-\frac{\ell}{2})}$. Using the second relation in (5.7), one obtains

$$\bar{\mathcal{D}}^-\Phi^{(N+\ell,-\frac{\ell}{2})} = -\ell\,\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\,. \tag{5.40}$$

For $\ell = 0$ this reproduces the first of the LLL supersphere wave function constraints (5.20b), while for $\ell \geq 1$ it yields the relation inverse to (5.35). So, at $\ell \neq 0$ the relation (5.35) is invertible (this property replaces the gauge invariance (5.25) of the LLL case).

Using (5.35) and (5.40), as well as the (anti)commutation relations (5.7), one can express both previously defined auxiliary functions (5.26) and (5.29) through the $(N - \frac{1}{2}, M = 0)$ superflag reduced wave function $\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$:

$$\begin{aligned} \Psi_\ell^{(N,0)} + \widehat{\Psi}_\ell^{(N,0)} \equiv \Psi_\ell^{(N)} &= \mathcal{D}^+ \left[(\mathcal{D}^{--})^\ell \tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\right] = \mathcal{D}^+ \tilde{\Psi}_\ell^{(N-\frac{1}{2},0)}\,, \\ \widehat{\Psi}_\ell^{(N,0)} &= -\ell\,\mathcal{D}^-(\mathcal{D}^{--})^{\ell-1}\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}\,. \end{aligned} \tag{5.41}$$

It is easy to show that the function $\Psi_\ell^{(N)}$ defined in (5.41) satisfies both the supersphere conditions (5.11). The first one is obeyed due to the property $\mathcal{D}^+\mathcal{D}^+ = 0$, while the second one due to chirality of $\tilde{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$. This property can be made manifest using the relation (5.40):

$$\Psi_\ell^{(N)} = -\frac{1}{\ell}\,(\mathcal{D}^+\bar{\mathcal{D}}^-)\Psi_\ell^{(N,0)} = \frac{1}{\ell}\,(\mathcal{D}^+\bar{\mathcal{D}}^-)\widehat{\Psi}_\ell^{(N,0)}\,. \tag{5.42}$$

Using the general transformation law (5.9) of the superflag superfields $\Psi^{(\hat{N},\hat{M})}$ under the odd SU(2|1) transformations, one immediately observes that the wave function $\Psi_\ell^{(N,0)}$ (5.41), as well as the LLL wave function $\Psi_0^{(N)}$ defined in (5.20), have the same transformation properties as the similar supersphere wave superfunctions defined in section 2.4 (eqs. (2.49) and (2.50)). On top of this, these constrained SU(2|1)/[U(1) × U(1)] superfields satisfy the basic condition (5.11), i.e. live on the supersphere, and are the eigenfunctions of the operator H (5.19) with the correct eigenvalues (2.33). As shown in the section 5.4, this 'would-be' Hamiltonian becomes exactly (2.27) when applied to these superfields, which may therefore be identified with the supersphere wave superfunctions defined in (2.32) and (2.37)–(2.39); this justifies the use of the same notation for both sets of superfields. In section 5.4 we shall also show how the reduced superfields $\Phi_\ell^{(\pm)}$ defined in (2.36) appear within the covariant SU(2|1)/[U(1) × U(1)] superfield approach.

Now let us discuss the relation between the SU(2|1) invariant integration measures on the supersphere and superflag. In accordance with the definitions (2.51), (2.52) and (3.30) they are

$$d\mu_{(SS)} = d\mu_0(K_2)^{-1}\,, \quad d\mu_{(SF)} = d\mu_0\partial_\xi\partial_{\bar{\xi}}(K_2)^{-2}\,. \tag{5.43}$$

Using

(i) the relations

$$\partial_\xi = K_2^{\frac{1}{2}}D^+\,, \quad \partial_{\bar{\xi}} = -K_2^{\frac{1}{2}}\bar{D}^-\,, \tag{5.44}$$

(ii) the fact that K_2 has no dependence on $\xi, \bar{\xi}$ and

(iii) that $\mathcal{D}^+, \bar{\mathcal{D}}^-$ have $\hat{B}$ charge zero, and assuming that the integrands in the corresponding integrals also have zero $\hat{B}$ charge, the measures in (5.43) are related by

$$d\mu_{(SF)} = d\mu_{(SS)}\bar{\mathcal{D}}^-\mathcal{D}^+\,. \tag{5.45}$$

Recalling the relations (5.22), (5.41) between the wave superfunctions of the supersphere and superflag models, which can be concisely written as

$$\Psi_{SS}^{(N)} = \mathcal{D}^+ \Psi_{SF}^{(N-\frac{1}{2},0)} , \quad \bar{\mathcal{D}}^- \Psi_{SF}^{(N-\frac{1}{2},0)} = \mathcal{D}^+ [\Psi_{SF}^{(N-\frac{1}{2},0)}]^* = 0 , \tag{5.46}$$

one gets the following simple relation between the inner products on the $M = 0$ superflag and supersphere:

$$\begin{aligned} \langle \Upsilon_{SF}^{(N-\frac{1}{2},0)} | \Psi_{SF}^{(N-\frac{1}{2},0)} \rangle &= \int d\mu_{(SF)} [\Upsilon_{SF}^{(N-\frac{1}{2},0)}]^* \Psi_{SF}^{(N-\frac{1}{2},0)} \\ &= \int d\mu_{(SS)} [\Upsilon_{SS}^{(N)}]^* \Psi_{SS}^{(N)} \equiv \langle \Upsilon_{SS}^{(N)} | \Psi_{SS}^{(N)} \rangle . \end{aligned} \tag{5.47}$$

This is the superfield form of the relation between the supersphere and superflag norms observed earlier at the component level.

It should be pointed out that the supersphere wave functions have zero norm with respect to the superflag inner product (this directly stems from (5.45) and the supersphere conditions (5.11)) but their norm is non-vanishing with respect to the supersphere inner product, i.e. when it is computed by the formula (5.47). Also, it is easy to check that any supersphere wave function is orthogonal to any $M = 0$ superflag wave function: their superflag inner products are vanishing. Thus the operator H of (5.19) has the unique normalizable LLL ground state with respect to the superflag inner product (recall that H is reduced to the superflag model Hamiltonian on the set of the covariantly chiral SU(2|1)/[U(1) × U(1)] superfields, eq. (5.24)). The supersphere wave function $\Psi_0^{(N)}$ has zero norm, and the possibility of adding to it the LLL ground state $M = 0$ superflag wave function provides the gauge invariance that is responsible for the fact that half of the component wave functions in the superflag LLL wave superfunction at $M = 0$ do not appear in its norm [2]. On the other hand, on the supersphere wave superfunctions the same operator H (5.19) is reduced to the supersphere Hamiltonian (2.27) (see section 5.4), with the *same* $\Psi_0^{(N)}$ as the LLL wave function. The latter has a non-zero norm with respect to the inner product on the supersphere.

To summarize, at given fixed $\hat{N} = N$, the $M = 0$ superflag model wave functions and the supersphere model wave functions span two different subspaces, closed under the action of SU(2|1), in the full variety of SU(2|1)/[U(1) × U(1)] superfields. These subspaces are orthogonal to each other with respect to the natural inner product on SU(2|1)/[U(1) × U(1)] . The supersphere wave functions have zero norm with respect to this product, but non-vanishing norm with respect to the inner product on the invariant submanifold SU(2|1)/ U(1|1) ⊂ SU(2|1)/[U(1) × U(1)]. The operator H is independently diagonalized on each of these two mutually orthogonal subspaces and is reduced on them, respectively, to the supersphere Hamiltonian (2.27) and to the $M = 0$ superflag Hamiltonian (5.38). Taken at the *same fixed* $\hat{N} = N$, these two models are not equivalent to each other. The N supersphere model is equivalent to the $M = 0$ superflag model with $N' = N - \frac{1}{2}$, and the covariant formulation given in this section makes this equivalence manifest.

5.3 Casimir considerations

To better understand the difference between the wave functions of the quantum supersphere and superflag models in the manifestly covariant unified description, let us compare the values of the Casimir operators (5.13) and (5.14) on these wave functions. The subsequent consideration is based on the fact that all covariant derivatives defined in (5.1), (5.3) *commute* with the Casimir operators C_2 and C_3 , so the values of the latter can be evaluated by applying them directly to the reduced wave functions.

For the general $M = 0$ superflag 'ground state' wave functions $\tilde{\Phi}^{(\tilde{N},\tilde{M})}$ subjected to the covariant chirality and analyticity conditions

$$\bar{\mathcal{D}}^{+}\tilde{\Phi}^{(\tilde{N},\tilde{M})} = \bar{\mathcal{D}}^{-}\tilde{\Phi}^{(\tilde{N},\tilde{M})} = \mathcal{D}^{++}\tilde{\Phi}^{(\tilde{N},\tilde{M})} = 0\,, \tag{5.48}$$

one finds

$$\begin{aligned} C_2 &= -2\tilde{M}[1 + 2(\tilde{M} + \tilde{N})]\,, \\ C_3 &= -2\tilde{M}[1 + 2(\tilde{M} + \tilde{N})][1 + 2(\tilde{N} + 2\tilde{M})] = [1 + 2(\tilde{N} + 2\tilde{M})]C_2\,. \end{aligned} \tag{5.49}$$

Both Casimirs vanish on the LLL state with $\tilde{N} = N, \tilde{M} = M = 0$, which corresponds to the 'atypical' representation of SU(2|1). For the ℓ-th LL 'ground state' with $\tilde{N} = N + \ell\,, \tilde{M} = -\frac{\ell}{2}$ the Casimirs take the values

$$C_2 = \ell(1 + \ell + 2N)\,, \quad C_3 = \ell(1 + 2N)(1 + \ell + 2N)\,. \tag{5.50}$$

For the general supersphere bosonic 'ground state' wave functions $\Phi^{(\tilde{N},\tilde{M})}$ subjected to another sort of Grassmann analyticity conditions, and the standard bosonic analyticity condition

$$\mathcal{D}^{+}\Phi^{(\tilde{N},\tilde{M})} = \bar{\mathcal{D}}^{+}\Phi^{(\tilde{N},\tilde{M})} = \mathcal{D}^{++}\Phi^{(\tilde{N},\tilde{M})} = 0\,, \tag{5.51}$$

one finds that

$$C_2 = -4\tilde{M}(\tilde{M} + \tilde{N})\,, \quad C_3 = -8\tilde{M}(\tilde{N} + 2\tilde{M})(\tilde{N} + \tilde{M}) = 2(\tilde{N} + 2\tilde{M})C_2\,. \tag{5.52}$$

On the supersphere LLL state with $\tilde{N} = N, \tilde{M} = 0$, these Casimirs again vanish, showing that this SU(2|1) multiplet is also 'atypical'. However, for any other LL with $\tilde{N} = N + \ell\,, \tilde{M} = -\frac{\ell}{2}$ the Casimirs take the values

$$C_2 = 2\ell\left(N + \frac{\ell}{2}\right), \quad C_3 = 4N\ell\left(N + \frac{\ell}{2}\right), \tag{5.53}$$

which does not coincide with (5.50) at the *same* fixed N. Thus in the superflag and supersphere cases at the same *fixed* N we deal with *different* representations of the supergroup SU(2|1). Comparing (5.50) and (5.52) one observes that the $M = 0$ superflag wave function backgrounds can be obtained from the supersphere ones by the substitution $N \to N + \frac{1}{2}$ in the latter. This correspondence just amounts to the equivalence of the N supersphere model and the $N' = N - \frac{1}{2}, M = 0$ superflag model, as established in the previous sections.

Finally, let us establish the precise relation between the operator C_2 (5.13) and the superflag model Hamiltonian in the covariant formulation, i.e. with

$$H_{SF} = -\mathcal{D}^{--}\mathcal{D}^{++} . \tag{5.54}$$

Using the properties that on the general superflag model superfields (5.10)

$$C_2 = H + (\hat{J}_3)^2 - (\hat{F})^2 = H + (N')^2 - (N' + 2M)^2 , \tag{5.55}$$

and, according to (5.24),

$$H = H_{SF} - 2M , \tag{5.56}$$

we find that $C_2 = H_{SF} - 2M\,(2M + 2N' + 1)$, in full agreement with eq. (4.27).

5.4 The supersphere model revisited

As the last topic of this section we establish the explicit relation with the consideration of section 2. Here we make use of the notation P to denote a point on the 'supersphere' superspace $(z, \bar{z}, \zeta, \bar{\zeta})$.

After a rather tedious calculation, the Hamiltonian operator (5.19), being applied to a general wave functions $\Psi^{(N,0)}$ defined by (5.8) with $N' = N, M' = 0$, can be cast in the following explicit form

$$\begin{aligned} H \,=\, & H_0 - \frac{\partial}{\partial \xi}\frac{\partial}{\partial \widehat{\bar{\xi}}} - K_2\left(\bar{\zeta} + \bar{z}\widehat{\bar{\xi}}\right)\left(\nabla^{(N)}_{\bar{z}} - \bar{\xi}\nabla^{(N)}_{\bar{\zeta}}\right)\frac{\partial}{\partial \widehat{\bar{\xi}}} \\ & + K_2^{-1}\bar{z}\left[(1 + z\bar{z})\,\xi + \bar{z}\zeta\right]\left(\nabla^{(N)}_{\bar{z}} - \bar{\xi}\nabla^{(N)}_{\bar{\zeta}}\right)\frac{\partial}{\partial \xi} \\ & - \left\{\bar{z}\left[K_1 - (\zeta\bar{\zeta})(\xi\bar{\xi})\right] - \xi\bar{\zeta}\right\}\nabla^{(N)}_{\bar{\zeta}}\,\frac{\partial}{\partial \xi} . \end{aligned} \tag{5.57}$$

Here $\widehat{\bar{\xi}}$ was already defined in (3.11)

$$\widehat{\bar{\xi}} = \bar{\xi}\,K_2 + \bar{\zeta}\,z , \tag{5.58}$$

and $\bar{\xi}$ is assumed to be expressed through $\widehat{\bar{\xi}}$ from (5.58),

$$\bar{\xi} = K_2^{-1}\left(\widehat{\bar{\xi}} - z\bar{\zeta}\right) . \tag{5.59}$$

The part H_0 coincides with the Hamiltonian (2.27):

$$H_0 = -g^{z\bar{z}}\nabla^{(N)}_z\nabla^{(N)}_{\bar{z}} - g^{\zeta\bar{\zeta}}\nabla^{(N)}_\zeta\nabla^{(N)}_{\bar{\zeta}} + g^{\zeta\bar{z}}\nabla^{(N)}_\zeta\nabla^{(N)}_{\bar{z}} - g^{z\bar{\zeta}}\nabla^{(N)}_z\nabla^{(N)}_{\bar{\zeta}} , \tag{5.60}$$

where the derivatives $\nabla^{(N)}_B, \nabla^{(N)}_{\bar{B}}$ were defined in (2.28), (2.30). The Hamiltonian H_0 contains no derivatives with respect to the Grassmann variables $\xi, \widehat{\bar{\xi}}$ which complement the supersphere to the superflag, so it is defined on the supersphere.

In terms of the covariant derivatives, the transition to the variable $\widehat{\bar{\xi}}$ of (5.58) eliminates the partial derivative with respect to $\bar{\xi}$ from the covariant derivative $\bar{\mathcal{D}}^{+}$. For what follows, it is instructive to give the expressions for the covariant derivatives $\bar{\mathcal{D}}^+, \mathcal{D}^+, \bar{\mathcal{D}}^-, \mathcal{D}^{\pm\pm}$ in the new basis and in application to the superfields with $\hat{J}_3 = 2N$ and $\hat{B} = 0$:

$$
\begin{aligned}
\bar{\mathcal{D}}^+ &= -(K_1K_2)^{\frac{1}{2}} \left\{\nabla^{(N)}_{\bar{\zeta}} - K_1^{-1}\left[(1+z\bar{z})\xi + \bar{z}\zeta\right]\left[\nabla^{(N)}_{\bar{z}} - \bar{\xi}\nabla^{(N)}_{\bar{\zeta}}\right]\right\},\\
\mathcal{D}^+ &= K_2^{-\frac{1}{2}}\partial_\xi\,, \qquad\qquad \bar{\mathcal{D}}^- = -K_2^{\frac{1}{2}}\partial_{\widehat{\bar{\xi}}}\,,\\
\mathcal{D}^{++} &= K_1^{\frac{1}{2}}K_2\left[\nabla^{(N)}_{\bar{z}} - \bar{\xi}\nabla^{(N)}_{\bar{\zeta}}\right],\\
\mathcal{D}^{--} &= K_1^{\frac{1}{2}}K_2\left\{\left[\nabla^{(N)}_{z} - \xi\nabla^{(N)}_{\zeta}\right] + K_2^{-1}\left[(1+\xi\widehat{\bar{\xi}})\bar{\zeta} + \bar{z}\widehat{\bar{\xi}}\right]\partial_{\widehat{\bar{\xi}}}\right\}.
\end{aligned}
\tag{5.61}
$$

Here $\bar{\xi}$ is assumed to be expressed as in (5.59).

Now, using these explicit expressions, one can show that the constraints (5.20) defining the LLL wave function $\Psi^{(N,0)}_0$ amount to the following explicit set of equations:

$$
\begin{aligned}
&(5.20\text{a}): \qquad \frac{\partial}{\partial\xi}\Psi^{(N,0)}_0 = \frac{\partial}{\partial\widehat{\bar{\xi}}}\Psi^{(N,0)}_0 = 0 \;\;\Rightarrow\;\; \Psi^{(N,0)}_0 = \Psi^{(N,0)}_0(P)\,,\\
&(5.20\text{b}): \qquad \nabla^{(N)}_{\bar{z}}\Psi^{(N,0)}_0 = \nabla^{(N)}_{\bar{\zeta}}\Psi^{(N,0)}_0 = 0\,.
\end{aligned}
\tag{5.62}
$$

Thus the LLL wave function in the 'superflag-inspired' formalism coincides with the LLL wave function $\Psi^{(N)}_0(P)$ defined by the constraints (2.32) of section 2: $\Psi^{(N,0)}_0 = \Psi^{(N)}_0$. The SU(2|1) transformation of $\Psi^{(N,0)}_0$ obtained by the general formula (5.9) coincides with the transformation law (2.49). The 'would-be' Hamiltonian (5.57) is reduced to the supersphere Hamiltonian H_0 on $\Psi^{(N,0)}_0$.

For the 'ground state' wave function $\Phi^{(N+\ell,-\frac{\ell}{2})}$ corresponding to the ℓ-th LL and satisfying the conditions (5.27), one is led to make the following redefinition

$$
\Phi^{(N+\ell,-\frac{\ell}{2})} = \left(1 - \xi\widehat{\bar{\xi}}\right)^{\ell} K_1^{-\frac{\ell}{2}} K_2^{-\ell}\Phi_{(\ell)}\,. \tag{5.63}
$$

The constraint (5.27a) then implies

$$
\frac{\partial}{\partial\xi}\Phi_{(\ell)} = 0 \;\;\Rightarrow \Phi_{(\ell)}(z,\bar{z},\zeta,\bar{\zeta},\widehat{\bar{\xi}}) = \omega_{(\ell)}(z,\bar{z},\zeta,\bar{\zeta}) + \widehat{\bar{\xi}}\,\phi_{(\ell)}(z,\bar{z},\zeta,\bar{\zeta})\,, \tag{5.64}
$$

while (5.27b) implies

$$
\begin{aligned}
\nabla^{(N)}_{\bar{z}}\Phi_{(\ell)} &= \nabla^{(N)}_{\bar{\zeta}}\Phi_{(\ell)} = 0 \;\;\Rightarrow\\
\nabla^{(N)}_{\bar{z}}\phi_{(\ell)} &= \nabla^{(N)}_{\bar{\zeta}}\phi_{(\ell)} = 0\,,\;\; \nabla^{(N)}_{\bar{z}}\omega_{(\ell)} = \nabla^{(N)}_{\bar{\zeta}}\omega_{(\ell)} = 0\,.
\end{aligned}
\tag{5.65}
$$

The covariantly chiral bosonic and fermionic functions $\phi_{(\ell)}$ and $\omega_{(\ell)}$ can be identified with the functions $\Phi^{(-)}_\ell$ and $\Phi^{(+)}_\ell$ defined by (2.34)–(2.36). Indeed, let us consider the transformation law of (5.63) under the odd SU(2|1) transformations. The left-hand side of (5.63) is transformed according to the general law (5.9), with $\hat{N} = N + \ell, \hat{M} = -\frac{\ell}{2}$. Then, using the coordinate transformations (2.10), (3.2) and (3.12), it is straightforward to find the transformation law of $\Phi_{(\ell)}(z,\bar{z},\zeta,\bar{\zeta},\widehat{\bar{\xi}})$:

$$
\delta\Phi_{(\ell)} = -\left[N\left(\epsilon^1\bar{\zeta} + \bar{\epsilon}_1\zeta\right) + \ell\left(\bar{\epsilon}_1\zeta - \epsilon^2\widehat{\bar{\xi}}\right)\right]\Phi_{(\ell)}. \tag{5.66}
$$

Recalling the transformation law of $\hat{\bar{\xi}}$, eq. (3.12),

$$\delta\hat{\bar{\xi}} = (\bar{\epsilon}_2 + z\bar{\epsilon}_1) - (\bar{\epsilon}_1\zeta)\hat{\bar{\xi}}\,, \tag{5.67}$$

and identifying

$$\omega_\ell = \Phi^{(+)}_\ell\,, \quad \phi_\ell = -\ell\,\Phi^{(-)}_\ell\,, \tag{5.68}$$

for the variations of the so defined $\Phi^{(\pm)}_\ell$ we obtain from (5.66) just the expressions (2.49).

As for the non-reduced H-eigenfunctions $\Psi^{(N,0)}_\ell$, $\widehat{\Psi}^{(N,0)}_\ell$ related to the 'ground state' ones by eqs. (5.26), (5.32), their relation to the functions $\Psi^{(N)}_{(+)\ell}(z,\bar{z},\zeta,\bar{\zeta})$, $\Psi^{(N)}_{(-)\ell}(z,\bar{z},\zeta,\bar{\zeta})$ used in section 2 is rather non-direct. We show this relation for the simplest $\ell = 1$ case. The detailed form of the relation between the wave functions $\Psi^{(N,0)}_1$ and $\Phi^{(N+1,-\frac{1}{2})}$ is as follows

$$\begin{aligned}\Psi^{(N,0)}_{\ell=1} = \mathcal{D}^{--}\Phi^{(N+1,-\frac{1}{2})} &= \left(1 - \xi\hat{\bar{\xi}}\right)\left[\nabla^{(N+1)}_z - \xi\nabla^{(N+1)}_\zeta\right]\Phi_{(1)} \\ &\quad + K_2^{-1}\left(\bar{\zeta} + \bar{z}\hat{\bar{\xi}}\right)\frac{\partial}{\partial\hat{\bar{\xi}}}\Phi_{(1)} - \bar{z}\,K_2^{-1}\,\Phi_{(1)}\,,\end{aligned} \tag{5.69}$$

where we made use of (5.63) for $\ell = 1$. To calculate $\widehat{\Psi}^{(N,0)}_1$, we make use of the explicit expression for the covariant bosonic function $\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})}$ defined in (5.29), (5.32) and related to $\Phi^{(N+\ell,-\frac{\ell}{2})}$ by (5.30). It reads

$$\begin{aligned}\widehat{\Phi}^{(N+\ell-\frac{1}{2},-\frac{\ell}{2})} &= \bar{\mathcal{D}}^{-}\,\Phi^{(N+\ell,-\frac{\ell}{2})} \\ &= -K_1^{-\frac{\ell}{2}}K_2^{-\ell+\frac{1}{2}}\frac{\partial}{\partial\hat{\bar{\xi}}}\left[\left(1 - \xi\hat{\bar{\xi}}\right)^\ell\Phi_{(\ell)}(z,\bar{z},\zeta,\bar{\zeta},\hat{\bar{\xi}})\right].\end{aligned} \tag{5.70}$$

Then we find

$$\begin{aligned}\widehat{\Psi}^{(N,0)}_{\ell=1} = \mathcal{D}^{-}\bar{\mathcal{D}}^{-}\Phi^{(N+1,-\frac{1}{2})} &= -\left[\nabla^{(N+1)}_\zeta + \hat{\bar{\xi}}\,\nabla^{(N+1)}_z\right]\left(\xi + \partial_{\hat{\bar{\xi}}}\right)\Phi_{(1)} \\ &\quad - K_2^{-1}\left(\bar{\zeta} + \bar{z}\hat{\bar{\xi}}\right)\frac{\partial}{\partial\hat{\bar{\xi}}}\Phi_{(1)} + \bar{z}\,K_2^{-1}\,\Phi_{(1)}\,.\end{aligned} \tag{5.71}$$

It is easy to check that, in the full agreement with the relations (5.41) and (5.42),

$$\begin{aligned}\Psi^{(N,0)}_{\ell=1} + \widehat{\Psi}^{(N,0)}_{\ell=1} &= \nabla^{(N+1)}_z\Phi^{(+)}_1 + \nabla^{(N+1)}_\zeta\Phi^{(-)}_1 = \Psi^{(N)}_1\,, \\ \Psi^{(N)}_1 &= -(\mathcal{D}^{+}\bar{\mathcal{D}}^{-})\Psi^{(N,0)}_{\ell=1} = \partial_\xi\partial_{\hat{\bar{\xi}}}\Psi^{(N,0)}_{\ell=1}\,.\end{aligned} \tag{5.72}$$

In a similar way, using e.g. eq. (5.42), one can find that $\Psi^{(N)}_\ell = \Psi^{(N,0)}_\ell + \widehat{\Psi}^{(N,0)}_\ell$ are expressed through $\Phi^{(\pm)}_\ell$ just according to (2.37), (2.39). Because $\Psi^{(N)}_\ell$ satisfy the conditions (5.11), they do not depend on $\xi, \hat{\bar{\xi}}$ and on them the operator (5.57) is reduced to H_0, i.e. to the supersphere Hamiltonian.

Note that the eigenvalue relation for $\ell = 1$

$$H\,\Psi^{(N,0)}_{\ell=1} = (2N+1)\,\Psi^{(N,0)}_{\ell=1}\,, \tag{5.73}$$

can be shown to imply the relation

$$H_0 \left[\nabla_z^{(N+1)} - \xi \nabla_\zeta^{(N+1)}\right] \Phi_{(1)}(z,\bar z,\zeta,\bar\zeta,\widehat{\bar\xi})$$
$$= (2N+1)\left[\nabla_z^{(N+1)} - \xi \nabla_\zeta^{(N+1)}\right] \Phi_{(1)}(z,\bar z,\zeta,\bar\zeta,\widehat{\bar\xi})\,. \tag{5.74}$$

Since H_0 contains no any derivatives with respect to $\xi, \widehat{\bar\xi}$, (5.74) amounts to

$$\begin{aligned} H_0\, \nabla_z^{(N+1)} \Phi_{\ell=1}^{(\pm)} &= (2N+1)\, \nabla_z^{(N+1)} \Phi_{\ell=1}^{(\pm)}\,, \\ H_0\, \nabla_\zeta^{(N+1)} \Phi_{\ell=1}^{(\pm)} &= (2N+1)\, \nabla_\zeta^{(N+1)} \Phi_{\ell=1}^{(\pm)}\,, \end{aligned} \tag{5.75}$$

where the functions $\Phi_\ell^{(\pm)}(z,\bar z,\zeta,\bar\zeta)$ are defined by the $\widehat{\bar\xi}$ expansion in (5.64) and by (5.68). Analogous relations can be obtained for $\ell > 1$. Such eigenfunctions have complicated SU(2|1) transformation laws and presumably correspond to some composite higher superspin SU(2|1) multiplets.

5.5 Digression: SU(2|1)/ U(2) superfields

For completeness, we comment here on another subclass of general superflag superfields: those that are defined on the purely fermionic coset space SU(2|1)/ U(2). These superfields do not depend on the coordinates z and $\bar z$, so they are defined by the following SU(2|1) covariant condition

$$\mathcal{D}^{++}\Phi^{(0,\hat M)} = \mathcal{D}^{--}\Phi^{(0,\hat M)} = 0\,. \tag{5.76}$$

By virtue of the last commutation relation in (5.7), these constraints are compatible only when $\hat J_3 = \hat N = 0$, in which case it is convenient to pass back to the coordinates ξ^1, ξ^2 defined in (3.19), and used in [2]. In these coordinates, $\mathcal{D}^{\pm\pm}$ involve only the partial derivatives ∂_z and $\partial_{\bar z}$, so it becomes manifest that the constraints (5.76) eliminate $z, \bar z$ dependence; i.e.

$$(5.76) \quad \Rightarrow \quad \Phi^{(0,\hat M)} = \Phi^{(0,\hat M)}(\xi^1,\xi^2,\bar\xi_1,\bar\xi_2)\,. \tag{5.77}$$

It is consistent to further impose on these general SU(2|1)/ U(2) superfields either the covariant chirality conditions

$$\bar{\mathcal{D}}^{+}\Phi_{(1)}^{(0,\hat M)} = \bar{\mathcal{D}}^{-}\Phi_{(1)}^{(0,\hat M)} = 0\,, \tag{5.78}$$

or the covariant anti-chirality conditions

$$\mathcal{D}^{+}\Phi_{(2)}^{(0,\hat M)} = \mathcal{D}^{-}\Phi_{(2)}^{(0,\hat M)} = 0\,. \tag{5.79}$$

These (anti)chirality constraints can be solved explicitly in terms of 'small-(anti)analytic' superfields, $\varphi_{(1)}$ or $\varphi_{(2)}$, which depend (anti)holomorphically on half of the fermionic coordinates; e.g.

$$\Phi_{(1)}^{(0,\hat M)} = (K_1)^{\hat M} \varphi_{(1)}^{(0,\hat M)}(\xi^1,\xi^2)\,, \qquad K_1 = 1 - \xi^1\bar\xi_1 - \xi^2\bar\xi_2\,. \tag{5.80}$$

Just this kind of superfield appeared as a wave superfunction in the model of odd SU(2|1) invariant quantum mechanics considered in [9]. There, the Lagrangian was taken to be the fermionic WZ term corresponding to $\mathrm{U}(1) \subset \mathrm{U}(2)$, so the Hamiltonian is zero and all states are described by a single chiral LLL wave superfunction. One could extend this model by adding to the WZ term the square of purely fermionic coset Cartan forms. In this case one should expect to have to consider higher Landau levels and the possibility of ghosts. The planar limit of such a model was studied in [4] under the rubric 'fermionic Landau model'; it was found that there are just two Landau levels in this limit, and that ghosts can be eliminated by an appropriate non-trivial choice of Hilbert space metric. We shall not pursue this investigation further here since the $\mathrm{SU}(2|1)/\,\mathrm{U}(2)$ Landau models cannot be considered as 'spherical' super-Landau models.

6. Conclusions

This paper concludes a series of earlier investigations into SU(2|1)-invariant extensions of the SU(2)-invariant spherical Landau models, parametrized by an integer electric charge $2N$. At the classical level, these models involve additional anti-commuting variables which, upon quantization, lead to additional quantum states in each Landau level such that each level furnishes a representation of the supergroup SU(2|1).

The series began with a study of the lowest Landau level for a particle on the supersphere $\mathbb{CP}^{(1|1)} \cong \mathrm{SU}(2|1)/\,\mathrm{U}(1|1)$, as a special case of $\mathbb{CP}^{(n|m)}$. One may take a limit in which only the lowest Landau level survives and in this limit the model provides a 'quantum superspace' description of the fuzzy superspheres of fuzzy degree $2N$ [1]. The quantum states of the lowest Landau level all have positive norm with respect to an SU(2|1)-invariant inner product that is naturally defined as a superspace integral, but this inner product implies the existence of negative norm states, or 'ghosts', in all higher levels. This unsatisfactory state of affairs is ameliorated in the 'superflag' Landau models which are based on the coset superspace $\mathrm{SU}(2|1)/[\mathrm{U}(1) \times \mathrm{U}(1)]$ and which involve an additional anti-commuting variable; these models also have an additional parameter, M, which has no effect on the energy levels but does have an effect on the norms of states [2]. For positive M it was found that the first $[2M]+1$ Landau levels are ghost-free, in the natural superspace norm, although there are still ghosts in higher Landau levels, and in all levels for $M < 0$.

An unusual feature of the superflag Landau models is that zero-norm states appear for non-negative integer $2M$. This is due to the existence, for non-negative M, of a fermionic gauge symmetry of the classical theory within the phase-space 'shell' of energy $2M$, which has an effect on the quantum theory when $2M$ is a non-negative integer. This unusual feature was investigated in detail in the context of the planar limit, which yields the 'planar superflag' Landau models [3]; in particular, it was shown that zero-norm states in the lowest Landau level of the $M = 0$ planar superflag Landau model ensure the equivalence of this model with the 'superplane' Landau model, obtained as the planar limit of the superspherical Landau model. The latter is very similar to a model studied earlier by Hasebe [7], but differs in the reality conditions imposed on the anti-commuting variables.

One surprising aspect of the superplane Landau model is that the energy spectrum is precisely that of a model of supersymmetric quantum mechanics, at least if one quantizes in such a way that the state space is a conventional Hilbert space and not a vector su-

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perspace. This feature implies the existence of an alternative positive norm, with respect to which the superplane Landau model is both unitary and 'worldline' supersymmetric (and this is implicit in Hasebe's work on his 'superplane' model) but it is not obvious that a positive norm will preserve the original 'internal' supersymmetry that motivated the model's construction. The planar super-Landau models were 'revisited' in [4] with the aim of clarifying this point. It was found that the 'internal' supersymmetry permits two possible norms, such that the alternative norm is positive when $M \leq 0$; a 'dynamical' combination of the two norms is needed for positivity when $M > 0$. A redefinition of the norm also changes the definition of hermitian conjugation, such that the new hermitian conjugates are 'shifted' by operators that generate 'hidden' symmetries. Remarkably, the non-zero 'shift' operators were found, for $M \leq 0$, to be the odd generators of a hidden worldline supersymmetry, spontaneously broken for $M < 0$ but unbroken for $M = 0$.

In this paper we have carried out a similar analysis for the superspherical Landau model, and for the associated superflag Landau models. One result of our analysis is the proof of a quantum equivalence between the $M = 0$ superflag Landau model with charge $2N' = 2N - 1$ and the superspherical Landau model with charge $2N$. Classically, there is an equivalence between these models for the *same* charge provided the energy is non-zero, so the 'quantum shift' of the charge by one unit is presumably due to some effect associated with zero energy configurations.

We have shown that SU(2|1) invariance of the general superflag model allows a positive Hilbert space norm that is a 'dynamical' combination of the 'naive' superspace norm and an 'alternative' norm that involves a non-trivial Hilbert space 'metric operator'. This alternative norm leads, by itself, to a unitary model when $-2N < 2M \leq 0$, and these are the cases that we have focused on. We have 'solved' these unitary models for all N: that is to say, we have found the complete SU(2|1) representation content at each Landau level. If it had been appreciated from the outset that SU(2|1) invariance is compatible with unitarity then it is possible that the superspherical Landau models would have been solved directly without the detour into the superflag Landau models, but the detour has proved instructive; the superflag models are simpler in some respects, and the superspherical models can be obtained by restricting to $M = 0$.

An interesting general issue, not investigated here, is how the semi-classical limit is modified by a change in the Hilbert space metric. In the coherent state approach to the classical limit, the symplectic 2-form associated to the classical dynamics clearly depends on the Hilbert space metric. A change from a non-positive metric to a positive one cannot be unitary, so we should expect a non-canonical transformation of the classical phase space. However, the negative norms that we find for the 'naive' Hilbert space metric are associated with the anti-commuting variables for which there is no truly classical limit, but this issue may be of interest in the context of the quasi- hermitian [15, 16] quantum mechanics, where it is well known that the non trivial Hilbert space metric plays a central role.

One of our objectives in this paper was to see whether the hidden worldline supersymmetry of the unitary planar super-Landau models is inherited from some analogous symmetry of unitary spherical super-Landau models. The introduction of a non-trivial 'metric operator', required to relate the alternative norm to the 'naive' one, implies the redefinition of some hermitian conjugates by 'shift' operators that are guaranteed by the formalism to be new 'hidden' symmetry generators. There is no guarantee that such 'hidden' symmetries will close to yield a finite-dimensional enlarged symmetry algebra but a closed subset can be found for $-2N < 2M < 0$. In these cases the manifest SU(2|1) symmetry is a subgroup of an SU(2|2) symmetry[10] with a central charge that is linear in the 'level operator'. The $M = 0$ case is similar in many respects but the lowest Landau level is now special and this prevents any simple construction of a finite basis of charges with level-independent (anti)commutation relations; it thus seems likely that any symmetry group of the superspherical Landau model that contains SU(2|1) but has higher dimension will have infinite dimension.[11]

Finally, one may hope that the unitary super-Landau models analyzed here and in our previous papers will find applications. One possibility is that they may provide an improved framework for the recently proposed [19] Landau-model approach to the Riemann hypothesis.

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A. CP^n Landau model

In this appendix we show how the method used in section 2.4 to solve the supersphere

[10] The supergroup SU(2|2) also arises in the context of integrable spin chains of relevance to the planar limit of $\mathcal{N} = 4$ super-Yang-Mills theory [17], but we are not aware of any connection to our work. More recently, SU(2|2) (actually, its some non-linear version) was identified as a 'hidden' symmetry of a model of $\mathcal{N} = 2$ supersymmetric Quantum Mechanics [18]; again, we are not aware of any relation to our work.

[11] This conclusion may be contrasted with claims made for the alternative OSp(1|2)-invariant 'superspherical' Landau model studied in [6], but any disagreement could be a consequence of a quantum inequivalence to the superspherical Landau model considered here.

model can be applied to the Landau model for a particle on $\mathbb{CP}^n$, which we view as a Kähler manifold of complex dimension n with isometry group $\mathrm{SU}(n+1)$. One may choose complex coordinates $\{z^a; a = 1, \dots, n\}$ such that the Kähler potential is

$$\mathcal{K} = \log\left(1 + \bar{z}\cdot z\right) , \qquad \left(\bar{z}\cdot z = \sum_{a=1}^{n} \bar{z}^a z^a\right) . \tag{A.1}$$

The corresponding Kähler metric is

$$g_{\bar{b}a} \equiv \partial_{\bar{b}}\partial_a \mathcal{K} = (1 + \bar{z}\cdot z)^{-1}\left[\delta_{ab} - (1 + \bar{z}z)^{-1} z^b \bar{z}^a\right] , \tag{A.2}$$

where we use the notation

$$\partial_a = \frac{\partial}{\partial z^a} , \qquad \partial_{\bar{a}} = \frac{\partial}{\partial \bar{z}^a} . \tag{A.3}$$

The Kähler potential $\mathcal{A}$ for the Kähler 2-form $\mathcal{F} = d\mathcal{A}$ is

$$\mathcal{A} \equiv -i\left(dz^a \partial_a - d\bar{z}^b \partial_{\bar{b}}\right)\mathcal{K} = dz^a \mathcal{A}_a + d\bar{z}^b \mathcal{A}_{\bar{b}} , \tag{A.4}$$

which gives

$$\mathcal{A}_a = -i\frac{\bar{z}^a}{1 + \bar{z}z} , \qquad \mathcal{A}_{\bar{b}} = i\frac{z^b}{1 + \bar{z}z} . \tag{A.5}$$

With these ingredients we may write down the classical Lagrangian for the $\mathbb{CP}^n$ Landau model:

$$L = \dot{z}^a \dot{\bar{z}}^b g_{\bar{b}a} + N\left(\dot{z}^a \mathcal{A}_a + \dot{\bar{z}}^b \mathcal{A}_{\bar{b}}\right) . \tag{A.6}$$

The infinitesimal $\mathrm{SU}(n+1)/\,\mathrm{U}(n)$ transformation of the coordinates is

$$\delta z^a = \varepsilon^a + (\bar{\varepsilon}\cdot z)\, z^a , \tag{A.7}$$

where $\{\varepsilon^a; a = 1, \dots, n\}$ are n constant complex parameters, and this induces the Kähler gauge transformation

$$\delta\mathcal{K} = (\bar{\varepsilon}\cdot z + \varepsilon\cdot\bar{z}) . \tag{A.8}$$

This is manifestly an infinitesimal isometry of the Kähler metric and a symmetry of $\mathcal{F}$. The $\mathrm{SU}(n+1)/\,\mathrm{U}(n)$ variation of the Lagrangian (A.6) is a total time derivative. The subgroup $\mathrm{U}(n) \subset \mathrm{SU}(n+1)$ is realized as linear transformations of $z^a, \bar{z}_a$.

With the standard notations for the conjugate momenta, one finds that the classical Hamiltonian is

$$H_{\mathrm{class}} = g^{a\bar{b}}\,(p_a - N\mathcal{A}_a)\,(\bar{p}_{\bar{b}} - N\mathcal{A}_{\bar{b}}) , \tag{A.9}$$

where the inverse metric is:

$$g^{a\bar{b}} = (1 + \bar{z}\cdot z)\left[\delta^{a\bar{b}} + z^a \bar{z}^b\right] . \tag{A.10}$$

We quantize the model via the standard replacement

$$p_a \to -i\partial_a , \qquad \bar{p}_{\bar{b}} \to -i\partial_{\bar{b}} . \tag{A.11}$$

Defining the quantum Hamiltonian through symmetric ordering of the covariant derivatives one has

$$H = -\frac{1}{2} g^{a\bar{b}} \left\{ \nabla_{\bar{b}}^{(N)}, \nabla_a^{(N)} \right\} = -g^{a\bar{b}} \nabla_a^{(N)} \nabla_{\bar{b}}^{(N)} + Nn \,, \tag{A.12}$$

where

$$\nabla_a^{(N)} = \partial_a - N\partial_a \mathcal{K} \,, \qquad \nabla_{\bar{b}}^{(N)} = \partial_{\bar{b}} + N\partial_{\bar{b}} \mathcal{K} \,. \tag{A.13}$$

These covariant derivatives have the commutation relation

$$\left[\nabla_{\bar{b}}^{(\tilde{N})}, \nabla_a^{(N)} \right] = -\left(N + \tilde{N} \right) g_{\bar{b}a} \,. \tag{A.14}$$

Now consider, for integer $2N \geq 0$, the sequence of wave functions

$$\Psi_\ell^{(N)} = \nabla_{a_1}^{(N+n+1)} \nabla_{a_2}^{(N+n+3)} \cdots \nabla_{a_\ell}^{(N+n+2\ell+1)} \Phi^{a_1 a_2 \cdots a_\ell} \,, \tag{A.15}$$

where $\Phi^{a_1 a_2 \cdots a_\ell}$ is a totally symmetric $(\ell, 0)$ tensor satisfying the analyticity conditions

$$\nabla_{\bar{b}}^{(N)} \Phi^{a_1 a_2 \cdots a_\ell} = 0 \,. \tag{A.16}$$

We claim that these are eigenfunctions of H with eigenvalue

$$E_\ell = \ell \left(2N + n + \ell \right) + Nn \,. \tag{A.17}$$

The proof goes very much like the one outlined in section 2.4 of the text, and rests on the identities

$$\begin{aligned} g^{a\bar{d}} \partial_c g_{\bar{d}b} &= -\left[1 + \bar{z} \cdot z \right]^{-1} \left[\delta_b^a \bar{z}^c + \delta_c^a \bar{z}^b \right] , \\ g^{d\bar{a}} \partial_{\bar{c}} g_{\bar{b}d} &= -\left[1 + \bar{z} \cdot z \right]^{-1} \left[\delta_b^a z^c + \delta_c^a z^b \right] . \end{aligned} \tag{A.18}$$

The total symmetry in the indices of $\Phi^{A_1 A_2 \cdots A_\ell}(P)$ is necessary in order to obtain the simple form of the 'semi-covariant' derivatives appearing in [A.15], from the fully covariant ones. Of course, cohomology arguments fix the value of N to be a half integer but this can also be deduced from convergence of the SU$(n+1)$-invariant norm

$$||\Psi||^2 = \int \prod_{a=1}^{n} dz^a d\bar{z}^a \, e^{-(n+1)\mathcal{K}} \, |\Psi|^2 \,. \tag{A.19}$$

One could pursue this analysis for $\mathbb{CP}^{(n|m)}$ but we will not attempt this here.

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